

# Extremes of local times for simple random walks on symmetric trees.

Yoshihiro Abe \*

## Abstract

We consider local times of the simple random walk on the  $b$ -ary tree of depth  $n$  and study a point process which encodes the location of the vertex with the maximal local time and the properly centered maximum over leaves of each subtree of depth  $r_n$  rooted at the  $(n - r_n)$  level, where  $(r_n)_{n \geq 1}$  satisfies  $\lim_{n \rightarrow \infty} r_n = \infty$  and  $\lim_{n \rightarrow \infty} r_n/n \in [0, 1)$ . We show that the point process weakly converges to a Cox process with intensity measure  $\alpha Z_\infty(dx) \otimes e^{-2\sqrt{\log b} y} dy$ , where  $\alpha > 0$  is a constant and  $Z_\infty$  is a random measure on  $[0, 1]$  which has the same law as the limit of the critical random multiplicative cascade measure up to a scale factor. As a corollary, we establish convergence in law of the maximum of the local times over leaves to a randomly shifted Gumble distribution.

*MSC 2010:* 60J55; 60J10; 60G70

*Keywords:* Local times; Simple random walk; Trees; Derivative martingale; Random multiplicative cascade measure.

## 1 Introduction

Much efforts have been made in the study of the so-called log-correlated random field such as the branching Brownian motion (BBM), the branching random walk (BRW), and the two-dimensional discrete Gaussian free field (DGFF). One of the remarkable features of these models is that laws of their maxima share common properties: each of the laws weakly converges to a randomly shifted Gumble distribution [30, 1, 16]. It is believed that each of the limiting extremal processes of a wide class of log-correlated fields converges to a so-called randomly shifted decorated Poisson point process [37] and it is established for the BBM [2, 5] and the BRW [33], and partially for the two-dimensional DGFF [11, 12].

It is well-known that local times of random walks on graphs have close relationships with DGFFs thanks to “the generalized second Ray-Knight theorem” [24] (this goes back to the Dynkin isomorphism [23]) which has many applications, for example, to the cover time [19, 18, 38]. Since the occupation time field of the simple random walk on the tree or on the two-dimensional lattice is closely related to the BRW or

---

\*Kyoto University, Kyoto 606-8502, Japan; yoshihiro@kurims.kyoto-u.ac.jp

two-dimensional DGFF respectively, it is natural to expect that their maxima and cover times belong to the universal class mentioned above: it is known that the cover times have subleading terms similarly to other log-correlated fields [21, 9], but further details are still open.

In this paper, we consider local times of the simple random walk on the  $b$ -ary tree of depth  $n$  at time much larger than the maximal hitting time and study convergence of a point process encoding extreme local maxima of the local times as  $n \rightarrow \infty$ .

To state our result, we begin with some notation. We fix an arbitrary integer  $b \geq 2$  throughout the paper. We will write  $T$  to denote the  $b$ -ary tree with root  $\rho$ : this is a rooted tree whose vertices have exactly  $b$  children. Let  $T_i$  be the  $i$ th generation of  $T$ . Set  $T_{\leq n} := \cup_{i=0}^n T_i$ . For  $v \in T$ , we will write  $|v|$  to denote the depth of  $v$ . For  $u \in T$ , let  $T^u$  be the subtree of  $T$  rooted at  $u$ , and we define  $T_i^u$  and  $T_{\leq n}^u$  similarly. For  $v, u \in T$ , let  $v \wedge u$  be the most recent common ancestor of  $v$  and  $u$ . Let  $X = (X_t, t \geq 0, P_v, v \in T_{\leq n})$  be the continuous-time simple random walk on  $T_{\leq n}$  with exponential holding times of parameter 1. We define the local time of  $X$  by

$$L_t^n(v) := \frac{1}{\deg(v)} \int_0^t 1_{\{X_s=v\}} ds, \quad v \in T_{\leq n}, t \geq 0,$$

where  $\deg(v)$  is the degree of  $v$ , and the inverse local time by

$$\tau(t) := \inf\{s \geq 0 : L_s^n(\rho) > t\}, \quad t \geq 0.$$

Let  $E(T)$  be the set of all edges on  $T$ . Let  $(Y_e)_{e \in E(T)}$  be independent and identically distributed random variables whose common law is the normal distribution with mean 0 and variance 1. To each  $v \in T$ , we assign  $h_v := \sum_{i=1}^{|v|} Y_{e_i^v}$ , where  $e_1^v, \dots, e_{|v|}^v$  are the edges on the unique shortest path from  $\rho$  to  $v$ . We will call  $(h_v)_{v \in T}$  a BRW on  $T$ . It is well-known that the so-called derivative martingale

$$D_n := \sum_{v \in T_n} \left( \sqrt{\log b} n - \frac{1}{\sqrt{2}} h_v \right) e^{-2\sqrt{\log b} \left( \sqrt{\log b} n - \frac{1}{\sqrt{2}} h_v \right)}$$

converges almost surely as  $n \rightarrow \infty$ , and the limit

$$D_\infty := \lim_{n \rightarrow \infty} D_n \tag{1.1}$$

is positive and finite almost surely (see, for example, [10, Theorem 5.1, 5.2] or [1, Proposition A.3]). To each  $v \in T$ , we assign a distinct label  $(\bar{v}_1, \dots, \bar{v}_{|v|})$  with  $\bar{v}_i \in \{0, \dots, b-1\}$  for all  $1 \leq i \leq |v|$  so that the vertices with labels  $(\bar{v}_1, \dots, \bar{v}_{|v|}, k)$ ,  $k \in \{0, \dots, b-1\}$  are children of  $v$ . We define the location of  $v \in T$  by

$$\sigma(v) := \sum_{i=1}^{|v|} \frac{\bar{v}_i}{b^i}. \tag{1.2}$$

For each  $n \in \mathbb{N}$  and  $x \in [0, 1]$ , let  $v(x)$  be the vertex in  $T_n$  with  $x \in [\sigma(v(x)), \sigma(v(x)) + b^{-n}]$ . (If we have two such vertices, we choose the one whose location is the largest.)

We define the random measure called a (critical) random multiplicative cascade measure by

$$Z_n(dx) := b^n \left( \sqrt{\log b} n - \frac{1}{\sqrt{2}} h_{v(x)} \right) e^{-2\sqrt{\log b} \left( \sqrt{\log b} n - \frac{1}{\sqrt{2}} h_{v(x)} \right)} dx, \quad (1.3)$$

where  $dx$  is the Lebesgue measure on  $[0, 1]$ . Barral, Rhodes, and Vargas [7] observed that

$$\text{the weak limit } Z_\infty := \lim_{n \rightarrow \infty} Z_n \text{ exists almost surely,} \quad (1.4)$$

and that  $Z_\infty$  satisfies that

$$\text{the law of } (Z_\infty(I_v))_{v \in T_n} \text{ is the same as that of } \left( e^{-2\sqrt{\log b} \left( \sqrt{\log b} n - \frac{1}{\sqrt{2}} h_v \right)} D_\infty^{(v)} \right)_{v \in T_n},$$

where  $I_v := [\sigma(v), \sigma(v) + b^{-n}]$  for each  $v \in T_n$ , and  $D_\infty^{(v)}, v \in T_n$  are independent copies of  $D_\infty$  which are independent of  $(h_v)_{v \in T_n}$ . See [8] for more details (for example, the KPZ formula) on  $Z_\infty$ . For each  $(x, y) \in [0, 1] \times \mathbb{R}$ , we write  $\delta_{(x,y)}$  to denote the Dirac measure at  $(x, y)$ . For each  $0 \leq m \leq n$ , we define the point process on  $[0, 1] \times \mathbb{R}$  by

$$\Xi_{n,t}^{(m)} := \sum_{u \in T_m} \delta \left( \sigma \left( \operatorname{argmax}_u L_{\tau(t)}^n \right), \max_{v \in T_{n-m}^u} \sqrt{L_{\tau(t)}^n(v)} - \sqrt{t} - a_n(t) \right), \quad (1.5)$$

where the centering sequence  $a_n(t)$  is given by

$$a_n(t) := \sqrt{\log b} n - \frac{3}{4\sqrt{\log b}} \log n - \frac{1}{4\sqrt{\log b}} \log \left( \frac{\sqrt{t} + n}{\sqrt{t}} \right), \quad (1.6)$$

and for each  $u \in T_m$ ,  $\operatorname{argmax}_u L_{\tau(t)}^n$  is the vertex  $v_*$  on  $T_{n-m}^u \subset T_n$  with  $L_{\tau(t)}^n(v_*) = \max_{v \in T_{n-m}^u} L_{\tau(t)}^n(v)$ . (If two or more vertices on  $T_{n-m}^u$  attain the maximum, we take the one whose location is the largest among such vertices.) We regard  $\Xi_{n,t}^{(m)}$  as an element of all Radon measures on Borel sets of  $[0, 1] \times \mathbb{R}$  topologized with the vague topology. Since this space is metrizable as a complete separable metric space, we can consider convergence in law of sequences of random measures. Given a random measure  $\nu$  on  $[0, 1] \times \mathbb{R}$ , we will write  $\text{PPP}(\nu)$  to denote a point process on  $[0, 1] \times \mathbb{R}$  which, conditioned on  $\nu$ , is a Poisson point process with intensity measure  $\nu$  (that is  $\text{PPP}(\nu)$  is a Cox process). We now state the main result of this paper:

**Theorem 1.1** *There exist  $c_1 > 0$  such that for any sequence  $(t_n)_{n \geq 1}$  with  $\lim_{n \rightarrow \infty} \frac{\sqrt{t_n}}{n} = \theta \in [0, \infty]$  and  $t_n \geq c_1 n \log n$  for each  $n \in \mathbb{N}$ , and any sequence  $(r_n)_{n \geq 1}$  with  $\lim_{n \rightarrow \infty} r_n = \infty$  and  $\lim_{n \rightarrow \infty} r_n/n \in [0, 1)$ , the point process  $\Xi_{n,t_n}^{(n-r_n)}$  converges in law to a Cox process*

$$\text{PPP} \left( \frac{4}{\sqrt{\pi}} \beta_* \gamma_* Z_\infty(dx) \otimes 2\sqrt{\log b} e^{-2\sqrt{\log b} y} dy \right) \quad (1.7)$$

as  $n \rightarrow \infty$ , where  $Z_\infty$  is the random measure on  $[0, 1]$  in (1.4),

$$\beta_* := \begin{cases} \sqrt{\frac{\theta+1}{\theta+\sqrt{\log b}}} & \text{if } \theta \in [0, \infty), \\ 1 & \text{if } \theta = \infty, \end{cases} \quad (1.8)$$

and

$$\gamma_* := \lim_{\ell \rightarrow \infty} \int_{\ell^{2/5}}^{\ell} z e^{2\sqrt{\log b} z} \mathbb{P} \left( \max_{v \in T_{\ell}} \frac{1}{\sqrt{2}} h_v > \sqrt{\log b} \ell + z \right) dz. \quad (1.9)$$

**Remark 1.2** *The existence of the limit (1.9) is non-trivial. It is proved in the proof of Proposition 5.1. Results similar to Theorem 1.1 are known for the BBM [4] and for the two-dimensional DGFF [11]. Our setting is inspired by [13]. The convergence of the full extremal process has been established for the BBM [2, 5] and the BRW [33]. Related convergence for the local times on the  $b$ -ary tree will be studied in a sequel paper.*

By Theorem 1.1 and a tail estimate of the maximum of local times over leaves (Proposition 3.1(i) below), we have:

**Corollary 1.3** *There exists  $c_1 > 0$  such that for all  $\lambda \in \mathbb{R}$  and any sequence  $(t_n)_{n \geq 1}$  with  $\lim_{n \rightarrow \infty} \frac{\sqrt{t_n}}{n} = \theta \in [0, \infty]$  and  $t_n \geq c_1 n \log n$  for each  $n \in \mathbb{N}$ ,*

$$\lim_{n \rightarrow \infty} P_{\rho} \left( \max_{v \in T_n} \sqrt{L_{\tau(t_n)}^n(v)} \leq \sqrt{t_n} + a_n(t_n) + \lambda \right) = \mathbb{E} \left[ e^{-\frac{4}{\sqrt{\pi}} \beta_* \gamma_* D_{\infty} e^{-2\sqrt{\log b} \lambda}} \right], \quad (1.10)$$

where  $D_{\infty}$ ,  $\beta_*$  and  $\gamma_*$  are given by (1.1), (1.8) and (1.9), respectively.

**Remark 1.4** *Let  $(h_v)_{v \in T}$  be a BRW on  $T$ . By Theorem 2.4(ii) and Lemma A.5, one can show that for all  $\lambda \in \mathbb{R}$  and any sequence  $(t_n)_{n \geq 1}$  with  $\lim_{n \rightarrow \infty} \sqrt{t_n}/n^2 = \infty$ ,*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P} \left( \max_{v \in T_n} \frac{1}{\sqrt{2}} h_v \leq m_n + \lambda \right) \\ &= \lim_{n \rightarrow \infty} P_{\rho} \left( \max_{v \in T_n} \sqrt{L_{\tau(t_n)}^n(v)} \leq \sqrt{t_n} + m_n + \lambda \right) = \mathbb{E} \left[ e^{-\frac{4}{\sqrt{\pi}} \gamma_* D_{\infty} e^{-2\sqrt{\log b} \lambda}} \right], \end{aligned} \quad (1.11)$$

where  $\gamma_*$  is given in (1.9) and the centering sequence  $m_n$  is defined by

$$m_n := \sqrt{\log b} n - \frac{3}{4\sqrt{\log b}} \log n.$$

(Note that the convergence of the maximum of the BRW has already been established in [6, 1, 17].) The centering sequence  $a_n(t_n)$  in (1.10) is different from  $m_n$  by the term  $\frac{1}{4\sqrt{\log b}} \log \left( \frac{\sqrt{t_n} + n}{\sqrt{t_n}} \right)$  which is non-negligible only when  $\theta < \infty$ .

The organization of the paper is as follows. Section 2 gives preliminary lemmas which we use repeatedly throughout the paper. In Section 3, we obtain tail probabilities of the maximum of local times over leaves which are essential to next sections. In Section 4, we show that two leaves with local times near maxima are either very close or far away. This implies that  $\Xi_{n,t_n}^{(q)}|_{[0,1] \times [z, \infty)} = \Xi_{n,t_n}^{(n-r_n)}|_{[0,1] \times [z, \infty)}$  with probability tending to 1 as  $n \rightarrow \infty$  and then  $q \rightarrow \infty$ , which is one of the key steps in the proof of Theorem 1.1. In Section 5, we obtain a limiting tail of the maximum of local times over leaves which is crucial to study the Laplace functional of  $\Xi_{n,t_n}^{(q)}$ . In Section 6, we give the proof of Theorem 1.1 and Corollary 1.3.

To prove Theorem 1.1, we use techniques developed in [22, 16] in the context of the two-dimensional DGFF. Since local times are not Gaussian random variables, we need some modifications in our setting. To overcome the difficulty, inspired by [21, 9], we apply the generalized second Ray-Knight theorem mentioned above. It implies that the local time process along the path from the root to a leaf behaves like a zero-dimensional squared Bessel process and this enables us to employ the argument of [22, 16]. We should emphasize that it is more convenient to study “continuous” version of local times rather than the original “discrete” ones especially when we estimate tail probabilities of the maximum of the local times over leaves. To take the advantage, motivated by [31, 38], we consider the local time process of the Brownian motion on the associated metric tree as the “continuous” version.

We will write  $c_1, c_2, \dots$  to denote positive universal constants whose values are fixed within each argument. We use  $c_1(M), c_2(M), \dots$  for positive constants which depend on  $M$ . Given sequences  $(c_n)_{n \geq 1}$  and  $(c'_n)_{n \geq 1}$ , we write  $c'_n = O(c_n)$  if there exists an universal constant  $C$  such that  $|c'_n/c_n| \leq C$  for all  $n \geq 1$ . We write  $|S|$  to denote the cardinality of a set  $S$ .

## 2 Preliminary lemmas

In this section, we collect some lemmas which we use repeatedly throughout the paper. We first recall the metric tree and the Brownian motions on it. In the study of local times of random walks on graphs, Lupu [31] and Zhai [38] found usefulness of the corresponding metric graphs and Brownian motions. We follow the approach and find its advantages in obtaining precise tail probabilities of the maximum of local times over leaves on the  $b$ -ary tree. Let  $T$  be the  $b$ -ary tree. We regard each  $e \in E(T)$  as an interval of length  $1/2$  by setting  $I_e := \{e\} \times (0, \frac{1}{2})$ . Set  $\bar{I}_e := I_e \cup \{e^-, e^+\}$ , where  $e^-, e^+ \in T$  be the endpoints of the edge  $e$ . Let  $\pi_e$  be the map from  $I_e$  to  $(0, \frac{1}{2})$  defined by  $\pi_e((e, x)) := x$ . We extend  $\pi_e$  to the map from  $\bar{I}_e$  to  $[0, \frac{1}{2}]$  by setting  $\pi_e(e^-) := 0, \pi_e(e^+) := \frac{1}{2}$ . We define a metric tree of depth  $n$  by

$$\tilde{T}_{\leq n} := T_{\leq n} \cup \bigcup_{e \in E(T_{\leq n})} I_e,$$

where  $E(T_{\leq n})$  is the set of all the edges in  $T_{\leq n}$ . For each  $k \in \mathbb{N}$  and  $v \in T$ , we will write  $\tilde{T}_{\leq k}^v$  to denote the metric tree corresponding to the subtree  $T_{\leq k}^v$ . We define the metric  $d(\cdot, \cdot)$  on  $\tilde{T}_{\leq n}$  as follows: for  $x, y \in \tilde{T}_{\leq n}$ , let  $e_x$  and  $e_y$  be the edges with  $x \in I_{e_x}$  and  $y \in I_{e_y}$ , respectively. In the case  $I_{e_x} \neq I_{e_y}$ , we set

$$d(x, y) := \min \left\{ |\pi_{e_x}(x) - \pi_{e_x}(v)| + \frac{1}{2} d_g(v, u) + |\pi_{e_y}(u) - \pi_{e_y}(y)| \right\},$$

where  $d_g$  is the graph distance on  $T_{\leq n}$  and the minimum is taken over  $v \in \{e_x^-, e_x^+\}$  and  $u \in \{e_y^-, e_y^+\}$ . In the case  $I_{e_x} = I_{e_y}$ , we set  $d(x, y) := |\pi_{e_x}(x) - \pi_{e_x}(y)|$ . We define a measure  $m$  on  $\tilde{T}_{\leq n}$  by

$$m(dx) := \sum_{e \in E(T_{\leq n})} 1_{I_e}(x) \nu_e(dx),$$

where  $v_e := v \circ \pi_e$ , and  $v$  is the Lebesgue measure on  $(0, 1/2)$ . We have a  $m$ -symmetric Hunt process on  $\tilde{T}_{\leq n}$  with continuous sample paths such that on each  $I_e$ , it behaves like a standard Brownian motion on  $(0, 1/2)$  until it hits  $\{e^-, e^+\}$ , and when it starts at a vertex  $v$ , it chooses one of the edges incident to  $v$  uniformly at random, and moves on it as described above. See, for example, [25, 29, 31] for the construction. We write  $\tilde{X} = (\tilde{X}_t, t \geq 0, \tilde{P}_x, x \in \tilde{T}_{\leq n})$  to denote the process and call it a Brownian motion on  $\tilde{T}_{\leq n}$ . It is known that  $\tilde{X}$  restricted on  $T_{\leq n}$  behaves like a simple random walk on  $T_{\leq n}$  in the following sense: for all  $v \in T_{\leq n}$  and  $1 \leq i \leq \deg(v)$ ,

$$\tilde{P}_v(\tilde{X}_{S_v} = v_i) = \frac{1}{\deg(v)}, \quad (2.1)$$

where  $v_1, \dots, v_{\deg(v)}$  are vertices on  $T_{\leq n}$  adjacent to  $v$ , and  $S_v$  is the hitting time of  $\{v_1, \dots, v_{\deg(v)}\}$  by  $\tilde{X}$ . See, for example, [25, Theorem 2.1] or [31, Section 2]. By [31, Section 2],  $\tilde{X}$  has a space-time continuous local time  $\{\tilde{L}_t^n(x) : (t, x) \in [0, \infty) \times \tilde{T}_{\leq n}\}$  and the following holds for each  $v \in T_{\leq n}$  under  $\tilde{P}_v$ :

$$\tilde{L}_{S_v}^n(v) \text{ has the same law as exponential distribution with mean } \frac{1}{\deg(v)}. \quad (2.2)$$

We define the inverse local time by

$$\tilde{\tau}(t) := \inf\{s \geq 0 : \tilde{L}_s^n(\rho) > t\}, \quad t > 0.$$

By (2.1) and (2.2), we have

$$\text{the law of } \left( \tilde{L}_{\tilde{\tau}(t)}^n(v) \right)_{v \in T_{\leq n}} \text{ under } \tilde{P}_\rho \text{ is the same as that of } \left( L_{\tau(t)}^n(v) \right)_{v \in T_{\leq n}} \text{ under } P_\rho. \quad (2.3)$$

The following is a remarkable property of local times of the Brownian motion on  $\tilde{T}_{\leq n}$ . The discrete version has already been considered in [18, Lemma 2.6].

**Lemma 2.1** *Fix  $n \in \mathbb{N}, t > 0$ , and  $a \in T_{\leq n} \setminus T_n$ . Let  $\mathcal{F}^\uparrow$  be the  $\sigma$ -field generated by  $\tilde{L}_{\tilde{\tau}(t)}^n(x)$ ,  $x \in \{a\} \cup \tilde{T}_{\leq n} \setminus \tilde{T}_{\leq n-|a|}^a$ . Then, the law of  $\{\tilde{L}_{\tilde{\tau}(t)}^n(x) : x \in \tilde{T}_{\leq n-|a|}^a \setminus \{a\}\}$  under  $\tilde{P}_\rho(\cdot | \mathcal{F}^\uparrow)$  is the same as that of  $\left\{ \tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(\tilde{L}_{\tilde{\tau}(t)}^n(a))^\downarrow(x) : x \in \tilde{T}_{\leq n-|a|}^a \setminus \{a\} \right\}$  under  $\tilde{P}_a$ , where  $\left\{ \tilde{L}_t^\downarrow(x) : (t, x) \in [0, \infty) \times \tilde{T}_{\leq n-|a|}^a \right\}$  is a local time of a Brownian motion on  $\tilde{T}_{\leq n-|a|}^a$  and  $\tilde{\tau}^\downarrow(t) := \inf\{s : \tilde{L}_s^\downarrow(a) > t\}$ .*

The proof of Lemma 2.1 is given in Section A.1.

Let  $\{\tilde{h}_x : x \in \tilde{T}_{\leq n}\}$  be a centered Gaussian process with  $\mathbb{E}(\tilde{h}_x \tilde{h}_y) = \tilde{E}_x(\tilde{L}_{H_y}^n(y))$  for all  $x, y \in \tilde{T}_{\leq n}$ , where  $H_x := \inf\{t \geq 0 : \tilde{X}_t = x\}$ . We will call  $\{\tilde{h}_x : x \in \tilde{T}_{\leq n}\}$  a Gaussian process associated with  $\tilde{T}_{\leq n}$ . We have an explicit representation of the covariance of  $\tilde{h}$ .

**Lemma 2.2** For all  $x, y \in \tilde{T}_{\leq n}$ ,

$$\mathbb{E}(\tilde{h}_x \tilde{h}_y) = d(\rho, x) + d(\rho, y) - d(x, y).$$

The proof of Lemma 2.2 is given in Section A.2. Let  $\{\tilde{h}_x : x \in \tilde{T}_{\leq n}\}$  be a Gaussian process associated with  $\tilde{T}_{\leq n}$  and  $(h_v)_{v \in T}$  be a BRW on  $T$ . By Lemma 2.2, we have for all  $n \in \mathbb{N}$

$$\{\tilde{h}_v : v \in T_{\leq n}\} = \{h_v : v \in T_{\leq n}\}, \text{ in law.} \quad (2.4)$$

Lemma 2.2 implies the following.

**Corollary 2.3** Let  $\{\tilde{h}_x : x \in \tilde{T}_{\leq n}\}$  be a Gaussian process associated with  $\tilde{T}_{\leq n}$ . For any leaf  $v \in T_n$ , the law of  $\{\tilde{h}_{v_s} : 0 \leq s \leq n\}$  is the same as that of a standard Brownian motion  $\{B_s : 0 \leq s \leq n\}$  on  $\mathbb{R}$ , where  $v_s$  is the point on the unique path in  $\tilde{T}_{\leq n}$  from  $\rho$  to  $v$  with  $d(\rho, v_s) = s/2$ .

The so-called generalized second Ray-Knight theorem connects the local time and the associated Gaussian process:

**Theorem 2.4** (i) ([24, Theorem 1]) Fix  $n \in \mathbb{N}$ . Let  $\{\tilde{h}_x : x \in \tilde{T}_{\leq n}\}$  be a Gaussian process associated with  $\tilde{T}_{\leq n}$ . For each  $t > 0$ , the law of  $\{\tilde{L}_{\tau(t)}^n(x) + \frac{1}{2}\tilde{h}_x^2 : x \in \tilde{T}_{\leq n}\}$  under  $\tilde{P}_\rho \times \mathbb{P}$  is the same as that of  $\{\frac{1}{2}(\tilde{h}_x + \sqrt{2t})^2 : x \in \tilde{T}_{\leq n}\}$ .

(ii) ([38]) For all  $t > 0$  and  $n \in \mathbb{N}$ , on the same probability space, one can construct a local time  $(L_{\tau(t)}^n(v))_{v \in T_{\leq n}}$  and two BRWs  $(h_v)_{v \in T_{\leq n}}$ ,  $(h'_v)_{v \in T_{\leq n}}$  on  $T_{\leq n}$  satisfying the following:

$$(L_{\tau(t)}^n(v))_{v \in T_{\leq n}} \text{ and } (h_v)_{v \in T_{\leq n}} \text{ are independent,} \quad (2.5)$$

$$L_{\tau(t)}^n(v) + \frac{1}{2}(h_v)^2 = \frac{1}{2}(h'_v + \sqrt{2t})^2, \text{ for each } v \in T_{\leq n}, \text{ almost surely.} \quad (2.6)$$

The construction of the coupling in Theorem 2.4(ii) can be found in the proof of Theorem 3.1 of [38]. Let  $C[0, \infty)$  be the space of real-valued continuous functions on  $[0, \infty)$  and  $\mathcal{B}(C[0, \infty))$  be the  $\sigma$ -field generated by cylinder sets in  $C[0, \infty)$ . We have a nice connection between the local time and the 0-dimensional squared Bessel process.

**Lemma 2.5** For all  $t > 0$  and  $v \in T_n$ , the law of  $\{\tilde{L}_{\tau(t)}^n(v_s) : 0 \leq s \leq n\}$  is the same as that of  $\{\frac{1}{2}X_s : 0 \leq s \leq n\}$  under  $\mathbb{Q}_{2t}^0$ , where  $\mathbb{Q}_x^d$  is a law on  $(C[0, \infty), \mathcal{B}(C[0, \infty)))$  under which the coordinate process  $\{X_s : s \geq 0\}$  is a  $d$ -dimensional squared Bessel process started at  $x$ .

The proof of Lemma 2.5 is based on an additivity property of squared Bessel processes, Corollary 2.3 and Theorem 2.4(i). See [9, Proof of Lemma 7.7] for the details. It is known that the laws of 0-dimensional and 1-dimensional squared Bessel processes are related to each other by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}_x^0}{d\mathbb{Q}_x^1} \Big|_{\mathcal{F}_t \cap \{H_0 > t\}} = \left(\frac{x}{X_t}\right)^{1/4} \exp\left(-\frac{3}{8} \int_0^t \frac{1}{X_s} ds\right), \quad (2.7)$$

for all  $t > 0$  and  $x > 0$ , where  $H_0 := \inf\{t \geq 0 : X_t = 0\}$  and  $\mathcal{F}_t$  is the  $\sigma$ -field generated by  $\{X_s : s \leq t\}$ . See, for example, [9, (7.30)]. The transition semigroup  $\{Q_t^0 : t \geq 0\}$  of a 0-dimensional squared Bessel process is given by

$$Q_t^0(x, \cdot) = \exp\left(-\frac{x}{2t}\right) \delta_0 + \tilde{Q}_t(x, \cdot), \quad x > 0, \quad (2.8)$$

where  $\delta_0$  is the Dirac measure at 0 and  $\tilde{Q}_t(x, \cdot)$  has the density

$$q_t^0(x, y) = \frac{1}{2t} \sqrt{\frac{x}{y}} \exp\left(-\frac{x+y}{2t}\right) I_1\left(\frac{\sqrt{xy}}{t}\right), \quad x, y \in (0, \infty), \quad (2.9)$$

where  $I_1(\cdot)$  is the modified Bessel function of the first kind

$$I_1(x) = \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{2k+1} \frac{1}{k!(k+1)!}. \quad (2.10)$$

We will use the following asymptotic behavior of  $I_1(\cdot)$ :

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}} (1 + O(1/z)), \quad \text{as } z \rightarrow \infty. \quad (2.11)$$

See, for example, [36, Chapter XI, §1] or [14, Section 2] for the details on squared Bessel processes. To estimate tail probabilities of the maximum of local times over leaves, we frequently use the following.

**Lemma 2.6** ([16, Lemma 3.6]) *Fix a constant  $C > 0$ . For  $z > 0$  and  $s > 0$ , set*

$$\begin{aligned} \mu_{s,z}(x) dx &:= P_0^B \left( \frac{B_s}{\sqrt{2}} \in dx, \frac{B_r}{\sqrt{2}} \leq z, \text{ for all } 0 \leq r \leq s \right) \\ &= \frac{1}{\sqrt{\pi s}} \left( e^{-\frac{x^2}{s}} - e^{-\frac{(2z-x)^2}{s}} \right) dx, \quad x \leq z, \end{aligned} \quad (2.12)$$

$$\mu_{s,z}^*(x) dx := P_0^B \left( \frac{B_s}{\sqrt{2}} \in dx, \frac{B_r}{\sqrt{2}} \leq z + z^{\frac{1}{20}} + C(r \wedge (s-r))^{\frac{1}{20}}, \text{ for all } 0 \leq r \leq s \right).$$

(i) *There exists  $c_1 > 0$  such that for any  $z > 1$ ,  $s > 0$ , and  $x \leq z + z^{\frac{1}{20}}$ ,*

$$\mu_{s,z}^*(x) \leq c_1 z (z + z^{\frac{1}{20}} - x) s^{-\frac{3}{2}} e^{-\frac{x^2}{s}}.$$

(ii) *There exists  $\delta_z$  with  $\lim_{z \rightarrow \infty} \delta_z = 0$  such that for all  $z > 1$ ,  $x \leq 0$ , and  $s \geq x^2 + z^2$ ,*

$$\mu_{s,z}^*(x) \leq (1 + \delta_z) \mu_{s,z}(x).$$

### 3 Tail of maximum of local time over leaves

The aim of this section is to obtain the following tail estimates of the maximum of local times of the simple random walk on the  $b$ -ary tree over leaves. Recall (1.6).



**Proposition 3.1** (i) *There exist  $c_1, c_2 \in (0, \infty)$  such that for all  $t > 0, y \geq 0$ , and  $n \in \mathbb{N}$ ,*

$$P_\rho \left( \max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \leq c_1(1+y)e^{-2\sqrt{\log b} y} e^{-c_2 \frac{y^2}{n}}. \quad (3.1)$$

(ii) *There exist  $c_3 > 0$  and  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,  $y \in [0, 2\sqrt{n}]$ , and  $t \geq n$ ,*

$$P_\rho \left( \max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \geq c_3(1+y)e^{-2\sqrt{\log b} y}. \quad (3.2)$$

Recall the notation  $v_s$  in Corollary 2.3. We first prove Theorem 3.1(i). Fix  $\kappa \in \left(\frac{1}{2\sqrt{\log b}}, \infty\right)$ . For  $y > 0$  and  $n \in \mathbb{N}$ , set

$$G_y^n(t) := \left\{ \text{there exist } v \in T_n \text{ and } 0 \leq s \leq n \text{ such that} \right. \\ \left. \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \geq \sqrt{t} + \frac{a_n(t)}{n}s + \kappa(\log(s \wedge (n-s)))_+ + y + 1 \right\}, \quad (3.3)$$

where  $c_+ := \max\{c, 0\}$ . We need the following.

**Lemma 3.2** *There exist  $c_1, c_2 \in (0, \infty)$  such that for all  $t > 0$ ,  $n \in \mathbb{N}$ , and  $y \geq 0$ ,*

$$\tilde{P}_\rho(G_y^n(t)) \leq c_1(1+y)e^{-2\sqrt{\log b} y} e^{-c_2 \frac{y^2}{n}}. \quad (3.4)$$

*Proof.* We first consider the case  $y > M$ , where  $M$  is sufficiently large constant. For a fixed  $\delta \in (0, 1)$ , set

$$g_{y,t,n}(s) := \sqrt{t} + \frac{a_n(t)}{n}s + \kappa(\log(s \wedge (n-s)))_+ + y + 1, \quad 0 \leq s \leq n, \quad (3.5)$$

$$m_{y,t,n}(j) := \sqrt{t} + \frac{a_n(t)}{n}j + \kappa \min_{j \leq r \leq j+1} (\log(r \wedge (n-r)))_+ + y + 1, \quad 0 \leq j \leq n-1.$$

Recall the probability measure  $\mathbb{Q}_x^d$  defined in Lemma 2.5 and set

$$\tau := \inf \left\{ s \geq 0 : \sqrt{X_s/2} \geq g_{y,t,n}(s) \right\},$$

where  $X$  is the coordinate process. By Lemma 2.5, we have

$$\begin{aligned} & \tilde{P}_\rho(G_y^n(t)) \\ & \leq \tilde{P}_\rho \left( \text{there exist } 0 \leq j \leq n-1, v \in T_{j+1}, \text{ and } s \in (j, j+1] \text{ such that} \right. \\ & \quad \left. \sqrt{\tilde{L}_{\tau(t)}^n(v_r)} < g_{y,t,n}(r), \text{ for all } 0 \leq r \leq j, \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \geq g_{y,t,n}(s) \right) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=0}^{n-1} b^{j+1} \mathbb{Q}_{2t}^0 \left( \tau \in (j, j+1], \sqrt{X_{j+1}/2} \geq \delta m_{y,t,n}(j) \right) \\
&+ \sum_{j=0}^{n-1} b^{j+1} \mathbb{Q}_{2t}^0 \left( \tau \in (j, j+1], \sqrt{X_{j+1}/2} < \delta m_{y,t,n}(j) \right) \\
&=: \sum_{j=0}^{n-1} b^{j+1} I_j^{(1)} + \sum_{j=0}^{n-1} b^{j+1} I_j^{(2)}.
\end{aligned} \tag{3.6}$$

Fix  $0 \leq j \leq n-1$ . We first estimate  $I_j^{(2)}$ . We have

$$\begin{aligned}
I_j^{(2)} &= \mathbb{Q}_{2t}^0 \left[ 1_{\{\tau \in (j, j+1)\}} \mathbb{Q}_{X_\tau}^0 \left( \sqrt{X_{j+1-\tau}/2} < \delta m_{y,t,n}(j) \right) \right] \\
&= \mathbb{Q}_{2t}^0 \left[ 1_{\{\tau \in (j, j+1)\}} \exp \left( -\frac{X_\tau}{2(j+1-\tau)} \right) \right] \\
&+ \mathbb{Q}_{2t}^0 \left[ 1_{\{\tau \in (j, j+1)\}} \int_0^{2\delta^2(m_{y,t,n}(j))^2} \frac{\sqrt{X_\tau/z}}{2(j+1-\tau)} e^{-\frac{X_\tau+z}{2(j+1-\tau)}} I_1 \left( \frac{\sqrt{X_\tau z}}{j+1-\tau} \right) dz \right] \\
&=: J_1 + J_2,
\end{aligned} \tag{3.7}$$

where we have used the strong Markov property of a 0-dimensional squared Bessel process in the first equality and (2.8)-(2.9) in the second equality. By the definition of  $\tau$ , we have

$$J_1 \leq \exp\{-(m_{y,t,n}(j))^2\}. \tag{3.8}$$

Assume that  $\tau \in (j, j+1)$ . Recall (2.10). If  $z \leq \frac{M(j+1-\tau)^2}{X_\tau}$ , then we have

$$I_1 \left( \frac{\sqrt{X_\tau z}}{j+1-\tau} \right) \leq \sum_{k=0}^{\infty} (\sqrt{M}/2)^{2k+1} \frac{1}{k!(k+1)!} \leq c_1(M). \tag{3.9}$$

If  $z > \frac{M(j+1-\tau)^2}{X_\tau}$ , then by (2.11) and the assumption that  $M$  is sufficiently large, we have

$$I_1 \left( \frac{\sqrt{X_\tau z}}{j+1-\tau} \right) \leq c_2 \frac{e^{\frac{\sqrt{X_\tau z}}{j+1-\tau}}}{\sqrt{\frac{\sqrt{X_\tau z}}{j+1-\tau}}}, \tag{3.10}$$

By (3.9) and (3.10), we have

$$J_2 \leq c_3(M) \left( \max_{j \leq r \leq j+1} g_{y,t,n}(r) \right) \exp\{-c_4(m_{y,t,n}(j))^2\}, \tag{3.11}$$

where we have used the inequality  $(j+1-\tau)^{-1/2} e^{-\frac{c}{j+1-\tau}} \leq e^{-c}$  if  $c > 1/2$ . By (3.8) and (3.11), we have

$$\sum_{j=0}^{n-1} b^{j+1} I_j^{(2)} \leq c_5(M) \sum_{j=0}^{n-1} b^{j+1} e^{-c_6(\sqrt{t}+y+1)^2} e^{-c_7(\sqrt{t}+y+1)j} \leq c_8(M) e^{-c_6(\sqrt{t}+y+1)^2}, \tag{3.12}$$

where in the last inequality, we have used the assumption that  $y > M$  and  $M$  is sufficiently large.

Next, we will estimate  $I_j^{(1)}$ . Fix  $1 \leq j \leq n-1$ . We have

$$\begin{aligned}
I_j^{(1)} &= \mathbb{Q}_{2t}^1 \left[ \left( \frac{2t}{X_{j+1}} \right)^{1/4} \exp \left( -\frac{3}{8} \int_0^{j+1} \frac{ds}{X_s} \right) 1_{\{\tau \in (j, j+1], \sqrt{X_{j+1}/2} \geq \delta m_{y,t,n}(j)\} \cap \{H_0 > j+1\}} \right] \\
&\leq \sqrt{\frac{\sqrt{t}}{\delta m_{y,t,n}(j)}} P_{\sqrt{2t}}^B \left( B_r / \sqrt{2} < g_{y,t,n}(r), \text{ for all } 0 \leq r \leq j, \right. \\
&\quad \left. B_s / \sqrt{2} \geq g_{y,t,n}(s), \text{ for some } j \leq s \leq j+1 \right) \\
&\leq \sqrt{\frac{\sqrt{t}}{\delta m_{y,t,n}(j)}} E_0^B \left[ 1_{\{B_r / \sqrt{2} < g_{y,t,n}(r) - \sqrt{t}, \text{ for all } 0 \leq r \leq j\}} \right. \\
&\quad \left. P_{B_j}^B \left( B_s / \sqrt{2} \geq g_{y,t,n}(j+s) - \sqrt{t}, \text{ for some } s \in [0, 1] \right) \right], \quad (3.13)
\end{aligned}$$

where we have used (2.7) in the first equality, the fact that the law of a 1-dimensional squared Bessel process is the same as that of a square of a standard Brownian motion on  $\mathbb{R}$  in the second inequality, and the translation invariance and Markov property of  $B$  in the last inequality.

Let  $\tilde{P}_j^B$  be the probability measure defined by

$$\tilde{P}_j^B(A) = E_0^B \left[ 1_A \exp \left\{ \frac{\sqrt{2}a_n(t)}{n} B_j - \frac{(a_n(t))^2}{n^2} j \right\} \right], \quad A \in \sigma(B_s : s \leq j). \quad (3.14)$$

By the Girsanov theorem, under  $\tilde{P}_j^B$ , the process

$$\left\{ \tilde{B}_s := B_s - \frac{\sqrt{2}a_n(t)}{n} s : 0 \leq s \leq j \right\} \quad (3.15)$$

is a standard Brownian motion on  $\mathbb{R}$  started at 0. Note that we have

$$g_{y,t,n}(r) - \sqrt{t} - \frac{a_n(t)}{n} r \leq y + c_9 + \kappa \log(j \wedge (n-j)) + c_{10}(r \wedge (j-r))^{1/20}, \quad 0 \leq r \leq j,$$

$$g_{y,t,n}(j+s) - \sqrt{t} - \frac{a_n(t)}{n} j \geq y - c_{11} + \kappa \log(j \wedge (n-j)), \quad 0 \leq s \leq 1.$$

By this and (3.13), we have

$$\begin{aligned}
&\sqrt{\frac{\delta m_{y,t,n}(j)}{\sqrt{t}}} I_j^{(1)} \\
&\leq \tilde{E}_j^B \left[ e^{-\frac{\sqrt{2}a_n(t)}{n} \tilde{B}_j - \frac{(a_n(t))^2}{n^2} j} 1_{\{\tilde{B}_r / \sqrt{2} < y + c_9 + \kappa \log(j \wedge (n-j)) + c_{10}(r \wedge (j-r))^{1/20}, \text{ for all } 0 \leq r \leq j\}} \right]
\end{aligned}$$

$$\begin{aligned}
& P_{\tilde{B}_j}^B \left( \max_{0 \leq s \leq 1} \frac{B_s}{\sqrt{2}} \geq y - c_{11} + \kappa \log(j \wedge (n-j)) \right) \Bigg] \\
& \leq c_{12} e^{-\frac{2a_n(t)}{n}y - \frac{(a_n(t))^2}{n^2}j} \left( 1 + \frac{y}{j+1} \right) (y + c_9 + \kappa \log(j \wedge (n-j))) \\
& \quad \cdot j^{-\frac{3}{2}} (j \wedge (n-j))^{-2\sqrt{\log b} \kappa} e^{-c_{13} \frac{y^2}{j+1}}, \tag{3.16}
\end{aligned}$$

where we have used Lemma 2.6(i) and the estimate

$$P_0^B \left( \max_{0 \leq s \leq 1} \frac{B_s}{\sqrt{2}} \geq \lambda \right) \leq e^{-\lambda^2}, \quad \text{for each } \lambda > 0 \tag{3.17}$$

(see, for example, [28, Chapter 2, (8.4)]) in the last inequality. Similarly, in the case  $j = 0$ , by (2.7) and (3.17), we have

$$I_0^{(1)} \leq c_{14} P_0^B \left( \max_{0 \leq s \leq 1} \frac{B_s}{\sqrt{2}} \geq y + 1 \right) \leq c_{14} e^{-(y+1)^2}. \tag{3.18}$$

Thus, by (3.16) and (3.18), we have

$$\sum_{j=0}^{n-1} b^{j+1} I_j^{(1)} \leq c_{15} (1+y) e^{-2\sqrt{\log b} y} e^{-c_{16} \frac{y^2}{n}}. \tag{3.19}$$

Thus, by (3.6), (3.12) and (3.19), we have (3.4) for  $y > M$ . For  $y \leq M$ , (3.4) holds if we take  $c_1$  in (3.4) sufficiently large depending on  $M$ .  $\square$

Using Lemma 3.2, we now prove Proposition 3.1(i):

*Proof of Proposition 3.1(i).* Recall (3.3) and (3.5). We have

$$P_\rho \left( \max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \leq \tilde{P}_\rho(\cup_{v \in T_n} E_v^n(t)) + \tilde{P}_\rho(G_y^n(t)), \tag{3.20}$$

where for each  $v \in T_n$ , we set

$$\begin{aligned}
E_v^n(t) &:= \left\{ \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \leq g_{y,t,n}(s), \text{ for all } 0 \leq s \leq n, \right. \\
&\quad \left. \sqrt{\tilde{L}_{\tau(t)}^n(v)} \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \right\}.
\end{aligned}$$

By using Lemma 2.6(i), 2.5 and (2.7) in a way similar to (3.13) and (3.16), we have

$$\tilde{P}_\rho(E_v^n(t)) \leq c_1 b^{-n} (1+y) e^{-2\sqrt{\log b} y} e^{-c_2 \frac{y^2}{n}}, \quad \text{for each } v \in T_n, \tag{3.21}$$

where we have used the inequality

$$g_{y,t,n}(s) - \sqrt{t} - \frac{a_n(t)}{n} s \leq y + 1 + c_3 \kappa (s \wedge (n-s))^{1/20}, \quad \text{for each } 0 \leq s \leq n.$$

Thus, by (3.20)-(3.21) and Lemma 3.2, we have (3.1).  $\square$

Next, we prove Proposition 3.1(ii). Fix  $\delta \in (0, 1)$ . For  $v \in T_n$ , set the event

$$A_v^n(t) := \left\{ \delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n}s + y + 1, \text{ for all } 0 \leq s \leq n, \right. \\ \left. \sqrt{\tilde{L}_{\tau(t)}^n(v)} \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \right\}. \quad (3.22)$$

To obtain Proposition 3.1(ii), we will apply the second moment method to  $\sum_{v \in T_n} 1_{A_v^n(t)}$ . We first need the following:

**Lemma 3.3** *There exist  $c_1 > 0$  and  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,  $y \in [0, 2\sqrt{n}]$ ,  $t \geq n$ , and  $v \in T_n$ ,*

$$\tilde{P}_\rho(A_v^n(t)) \geq c_1 b^{-n} (1+y) e^{-2\sqrt{\log b} y}. \quad (3.23)$$

*Proof.* By Lemma 2.5 and (2.7), we have

$$\begin{aligned} & \tilde{P}_\rho(A_v^n(t)) \\ & \geq c_1 \sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y + 1}} P_0^B \left( -(1-\delta)\sqrt{t} \leq B_s/\sqrt{2} < \frac{a_n(t)}{n}s + y + 1, \right. \\ & \quad \left. \text{for all } 0 \leq s \leq n, B_n/\sqrt{2} \in [a_n(t) + y, a_n(t) + y + 1] \right) \\ & \geq c_1 \sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y + 1}} (J_1 - J_2), \end{aligned} \quad (3.24)$$

where we have used  $t \geq n$  to show that  $\exp\left(-\frac{3}{8} \int_0^n \frac{ds}{X_s}\right) \geq c_1$  under the event that  $\sqrt{X_s/2} \geq \delta\sqrt{t}$  for all  $0 \leq s \leq n$ , and set

$$J_1 := P_0^B \left( \frac{B_s}{\sqrt{2}} < \frac{a_n(t)}{n}s + y + 1, \text{ for all } 0 \leq s \leq n, \frac{B_n}{\sqrt{2}} \in [a_n(t) + y, a_n(t) + y + 1] \right),$$

$$J_2 := P_0^B \left( \frac{B_s}{\sqrt{2}} < -(1-\delta)\sqrt{t}, \text{ for some } 0 \leq s \leq n, \frac{B_n}{\sqrt{2}} \in [a_n(t) + y, a_n(t) + y + 1] \right).$$

We first obtain an upper bound of  $J_2$ . By using the density

$$P_0^B \left( B_s \in dx, \max_{0 \leq r \leq s} B_r \in dz \right) = \frac{2(2z-x)}{\sqrt{2\pi s^3}} e^{-\frac{(2z-x)^2}{2s}} dx dz, \quad s > 0, x \leq z, z \geq 0, \quad (3.25)$$

(see, for example, [28, Chapter 2, Proposition 8.1]), for all  $n \in \mathbb{N}$  and  $y \in [0, 2\sqrt{n}]$ , we have

$$J_2 = P_0^B \left( \max_{0 \leq s \leq n} B_s/\sqrt{2} > (1-\delta)\sqrt{t}, B_n/\sqrt{2} \in (-a_n(t) - y - 1, -a_n(t) - y] \right)$$

$$\leq c_2 b^{-n} n e^{-c_3 \sqrt{n}} \sqrt{\frac{\sqrt{t}+n}{\sqrt{t}}} e^{-2\sqrt{\log b} y}, \quad (3.26)$$

where we have used the symmetry of  $B$  in the first equality.

Next, we obtain a lower bound of  $J_1$ . Recall the probability measure  $\tilde{P}_n^B$  and the process  $\tilde{B}$  defined in (3.14) and (3.15). For all  $n \geq n_0$  ( $n_0$  is sufficiently large) and  $y \in [0, 2\sqrt{n}]$ , we have

$$\begin{aligned} J_1 &= \tilde{E}_n \left[ e^{-\frac{\sqrt{2}a_n(t)}{n}\tilde{B}_n - \frac{(a_n(t))^2}{n}} 1_{\{\tilde{B}_s/\sqrt{2} < y+1, \text{ for all } 0 \leq s \leq n, \tilde{B}_n/\sqrt{2} \in [y, y+1)\}} \right] \\ &\geq c_4 b^{-n} \sqrt{\frac{\sqrt{t}+n}{\sqrt{t}}} (y+1) e^{-2\sqrt{\log b} y}, \end{aligned} \quad (3.27)$$

where we used (2.12) and  $e^{-\frac{(y+1-z)^2}{n}} - e^{-\frac{(y+1+z)^2}{n}} \geq c_5 \frac{y+1}{n}$  for all  $z \in [1/2, 1]$  in the last inequality. Thus, by (3.24), (3.26)-(3.27), we have (3.23).  $\square$

To obtain upper bounds of  $\tilde{P}_\rho(A_u^n(t) \cap A_v^n(t)), u, v \in T_n$ , we need the following:

**Lemma 3.4** *There exists  $c_1 > 0$  such that for all  $n \in \mathbb{N}$ ,  $t > 0$ ,  $v \in T_n$ ,  $0 \leq \ell \leq n-1$ ,  $s < (\sqrt{t} + \frac{a_n(t)}{n}\ell + y + 1)^2$ , and  $y \geq 0$ ,*

$$\begin{aligned} \tilde{P}_{v_\ell} \left( \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(s)}^\downarrow}(v_r) < \sqrt{t} + \frac{a_n(t)}{n}r + y + 1, \text{ for all } \ell \leq r \leq n, \right. \\ \left. \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(s)}^\downarrow}(v) \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1) \right) \\ \leq c_1 (n-\ell)^{-3/2} \sqrt{\frac{\sqrt{s}}{\sqrt{t} + a_n(t) + y}} \left( \sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n}\ell + y + 1 \right) e^{-\frac{(\sqrt{t} - \sqrt{s} + a_n(t) + y)^2}{n-\ell}}, \end{aligned} \quad (3.28)$$

where  $\{\tilde{L}_r^\downarrow(x) : (r, x) \in [0, \infty) \times \tilde{T}_{\leq n-\ell}^{v_\ell}\}$  is a local time of a Brownian motion on  $\tilde{T}_{\leq n-\ell}^{v_\ell}$  and  $\tilde{\tau}^\downarrow(s) := \inf\{r \geq 0 : \tilde{L}_r^\downarrow(v_\ell) > s\}$ .

*Proof.* Let  $P_\ell$  be the probability on the left-hand side of (3.28). Recall the probability measure  $\tilde{P}_{n-\ell}^B$  and the process  $\tilde{B}$  defined in (3.14) and (3.15). By using Lemma 2.5 and (2.7) in a way similar to the proof of Proposition 3.1(i), we have

$$\begin{aligned} &\sqrt{\frac{\sqrt{t} + a_n(t) + y}{\sqrt{s}}} P_\ell \\ &\leq \tilde{E}_{n-\ell}^B \left[ e^{-\frac{\sqrt{2}a_n(t)}{n}\tilde{B}_{n-\ell} - \frac{(a_n(t))^2}{n^2}(n-\ell)} 1_{\left\{ \frac{\tilde{B}_{n-\ell}}{\sqrt{2}} \in \sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n}\ell + y + [0, 1) \right\}} \right. \\ &\quad \left. 1_{\left\{ \frac{\tilde{B}_r}{\sqrt{2}} < \sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n}\ell + y + 1, \text{ for all } 0 \leq r \leq n-\ell \right\}} \right] \end{aligned}$$

$$\leq c_1(n-\ell)^{-3/2} \left( \sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n} \ell + y + 1 \right) e^{-\frac{(\sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n} \ell + y)^2}{n-\ell}}, \quad (3.29)$$

where we have used the density (2.12) and the inequality  $1 - e^{-x} \leq x$  for each  $x \geq 0$  in the last inequality. Thus, we have obtained (3.28).  $\square$

By Lemma 3.4 with  $s = t$  and  $\ell = 0$ , we have the following.

**Corollary 3.5** *There exists  $c_1 > 0$  such that for all  $n \in \mathbb{N}$ ,  $t > 0$ ,  $v \in T_n$ , and  $y \geq 0$ ,*

$$\tilde{P}_\rho(A_v^n(t)) \leq c_1 b^{-n} (y+1) e^{-2\sqrt{\log b} y}. \quad (3.30)$$

*Proof of Proposition 3.1(ii).* Fix any  $n \geq n_0$ ,  $t \geq n$  and  $y \in [0, 2\sqrt{n}]$ , where we take  $n_0 \in \mathbb{N}$  large enough. Set

$$Z := \sum_{v \in T_n} 1_{A_v^n(t)}.$$

We have

$$P_\rho \left( \max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \geq \tilde{P}_\rho(Z \geq 1) \geq \frac{\left( \tilde{E}_\rho[Z] \right)^2}{\tilde{E}_\rho[Z^2]}. \quad (3.31)$$

By Lemma 3.3, we have

$$\tilde{E}_\rho[Z] \geq c_1(1+y) e^{-2\sqrt{\log b} y}. \quad (3.32)$$

The rest of the proof focuses on obtaining an upper bound of  $\tilde{E}_\rho[Z^2]$ . We have

$$\tilde{E}_\rho[Z^2] = \tilde{E}_\rho[Z] + \sum_{\ell=0}^{n-1} \sum_{\substack{v, u \in T_n, \\ |v \wedge u| = \ell}} \tilde{P}_\rho(A_v^n(t) \cap A_u^n(t)). \quad (3.33)$$

By Corollary 3.5, we have

$$\tilde{E}_\rho[Z] \leq c_2(y+1) e^{-2\sqrt{\log b} y}. \quad (3.34)$$

Fix  $1 \leq \ell \leq n-1$  and  $v, u \in T_n$  with  $|v \wedge u| = \ell$ . Let  $\left\{ \tilde{L}_s^\downarrow(x) : (s, x) \in [0, \infty) \times \tilde{T}_{\leq n-\ell}^{v_\ell} \right\}$  be a local time of a Brownian motion on  $\tilde{T}_{\leq n-\ell}^{v_\ell}$ . Set  $\tilde{\tau}(t) := \inf\{s \geq 0 : \tilde{L}_s^\downarrow(v_\ell) > t\}$ . For  $w \in \{v, u\}$  and  $s \geq 0$ , we define the event  $C_w^\downarrow(s)$  by

$$C_w^\downarrow(s) := \left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(s)}^\downarrow(w_r)} < \sqrt{t} + \frac{a_n(t)}{n} r + y + 1, \text{ for all } \ell \leq r \leq n, \right. \\ \left. \sqrt{\tilde{L}_{\tilde{\tau}(s)}^\downarrow(w)} \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \right\}.$$

By Lemma 2.1, we have

$$\tilde{P}_\rho(A_v^n(t) \cap A_u^n(t))$$

$$\begin{aligned}
&= \tilde{E}_\rho \left[ 1_{\left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n} s + y + 1, \text{ for all } 0 \leq s \leq \ell \right\}} \tilde{P}_{v_\ell} \left( \bigcap_{w \in \{v, u\}} C_w^\downarrow \left( \tilde{L}_{\tilde{\tau}(t)}^n(v_\ell) \right) \right) \right] \\
&= \tilde{E}_\rho \left[ 1_{\left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n} s + y + 1, \text{ for all } 0 \leq s \leq \ell \right\}} \prod_{w \in \{v, u\}} \tilde{P}_{v_\ell} \left( C_w^\downarrow \left( \tilde{L}_{\tilde{\tau}(t)}^n(v_\ell) \right) \right) \right] \\
&\leq \sum_{i=0}^{\lceil (1-\delta)\sqrt{t} + \frac{a_n(t)}{n} \ell + y \rceil} \tilde{E}_\rho \left[ 1_{\left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n} s + y + 1, \text{ for all } 0 \leq s \leq \ell \right\}} \right. \\
&\quad \left. 1_{\left\{ \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_\ell)} \in \sqrt{t} + \frac{a_n(t)}{n} \ell + y + 1 + [-i-1, -i] \right\}} \prod_{w \in \{v, u\}} \tilde{P}_{v_\ell} \left( C_w^\downarrow \left( \tilde{L}_{\tilde{\tau}(t)}^n(v_\ell) \right) \right) \right], \tag{3.35}
\end{aligned}$$

where we have used the independence of  $C_v^\downarrow(s)$  and  $C_u^\downarrow(s)$  for each  $s \geq 0$ . The independence follows from that of two types of excursions of a Brownian motion around  $v_\ell$  on  $\tilde{T}_{\leq n-\ell-1}^{v_{\ell+1}} \cup I_{\{v_\ell, v_{\ell+1}\}}$  or on  $\tilde{T}_{\leq n-\ell-1}^{u_{\ell+1}} \cup I_{\{v_\ell, u_{\ell+1}\}}$ . Fix  $i \leq \lceil (1-\delta)\sqrt{t} + \frac{a_n(t)}{n} \ell + y \rceil$ . By Lemma 3.4, under the event  $\left\{ \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_\ell)} \in \sqrt{t} + \frac{a_n(t)}{n} \ell + y + 1 + [-i-1, -i] \right\}$ , we have for all  $w \in \{v, u\}$

$$\tilde{P}_{v_\ell} \left( C_w^\downarrow \left( \tilde{L}_{\tilde{\tau}(t)}^n(v_\ell) \right) \right) \leq c_3 (n-\ell)^{-\frac{3}{2}} (i+1) \sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n} \ell + y + 1 - i}{\sqrt{t} + a_n(t) + y}} e^{-\frac{\left( \frac{a_n(t)}{n} (n-\ell) + i - 1 \right)^2}{n-\ell}}. \tag{3.36}$$

By almost the same argument as the proof of Lemma 3.4, we have

$$\begin{aligned}
&\sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n} \ell + y - i}{\sqrt{t}}} \tilde{P}_\rho \left( \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n} s + y + 1, \text{ for all } 0 \leq s \leq \ell, \right. \\
&\quad \left. \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_\ell)} \in \sqrt{t} + \frac{a_n(t)}{n} \ell + y + 1 + [-i-1, -i] \right) \\
&\leq c_4 b^{-\ell} \ell^{-3/2} (i+1) (y+1) e^{-2\sqrt{\log b} y} e^{\frac{2a_n(t)}{n} i} e^{\frac{3 \log n}{2n} \ell} e^{\frac{\log \left( \frac{\sqrt{t}+n}{\sqrt{t}} \right)}{2n} \ell}. \tag{3.37}
\end{aligned}$$

By (3.35)-(3.37), we have

$$\begin{aligned}
&\tilde{P}_\rho(A_v^n(t) \cap A_u^n(t)) \\
&\leq c_5 b^{-2n+\ell} \ell^{-3/2} (n-\ell)^{-3} n^3 (y+1) e^{-2\sqrt{\log b} y} e^{-\frac{3 \log n}{2n} \ell} e^{-\frac{\log \left( \frac{\sqrt{t}+n}{\sqrt{t}} \right)}{2n} \ell} \sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n} \ell + y}{\sqrt{t}}}. \tag{3.38}
\end{aligned}$$

By (3.38), we have

$$\sum_{\ell=1}^{n-1} \sum_{\substack{v, u \in T_n, \\ |v \wedge u| = \ell}} \tilde{P}_\rho(A_v^n(t) \cap A_u^n(t)) \leq c_6 (y+1) e^{-2\sqrt{\log b} y}. \tag{3.39}$$



In the case  $\ell = 0$ , by Corollary 3.5, we have

$$\sum_{\substack{v,u \in T_n, \\ |v \wedge u| = 0}} \tilde{P}_\rho(A_v^n(t) \cap A_u^n(t)) = \sum_{\substack{v,u \in T_n, \\ |v \wedge u| = 0}} \tilde{P}_\rho(A_v^n(t)) \tilde{P}_\rho(A_u^n(t)) \leq c_7(y+1)e^{-2\sqrt{\log b} y}, \quad (3.40)$$

where we have used the independence of  $A_v^n(t)$  and  $A_u^n(t)$  for each  $v, u \in T_n$  with  $|v \wedge u| = 0$  in the first equality. The independence follows from that of two types of excursions of a Brownian motion on  $\tilde{T}_{\leq n}$  around  $\rho$  restricted on  $\tilde{T}_{\leq n-1}^{v_1} \cup I_{\{\rho, v_1\}}$  or on  $\tilde{T}_{\leq n-1}^{u_1} \cup I_{\{\rho, u_1\}}$ . Thus, by (3.31)-(3.34) and (3.39)-(3.40), we have (3.2).  $\square$

## 4 Geometry of near maxima

In this section, we will prove that two leaves with local times near maxima are either very close or far away. More specifically, the following is the aim of this section.

**Proposition 4.1** *There exist  $c_1, c_2 \in (0, \infty)$ ,  $n_0, r_0 \in \mathbb{N}$ , and  $t_0 > 0$  such that for all  $n \geq n_0$ ,  $t \geq t_0$ , and  $r_0 \leq r \leq n/4$ ,*

$$P_\rho \left( \text{there exist } v, u \in T_n \text{ with } r \leq |v \wedge u| \leq n-r \text{ such that} \right. \\ \left. \sqrt{L_{\tau(t)}^n(v)}, \sqrt{L_{\tau(t)}^n(u)} \in [\sqrt{t} + a_n(t) - c_1 \log r, \infty) \right) \leq c_2 r^{-1/8}.$$

**Remark 4.2** *Results similar to Proposition 4.1 are known for the BBM [3] and for the two-dimensional DGFF [22].*

For  $n \in \mathbb{N}$ ,  $t > 0$ , and  $k \in \mathbb{Z}$ , set

$$\Gamma_k^n(t) := \left\{ v \in T_n : \sqrt{L_{\tau(t)}^n(v)} \in [\sqrt{t} + a_n(t) - k - 1, \sqrt{t} + a_n(t) - k] \right\}.$$

**Remark 4.3** *Fix  $n' > n \geq 1$ ,  $t > 0$ , and  $k \in \mathbb{Z}$ . Set*

$$\Gamma_k^{n',n}(t) := \left\{ v \in T_n : \sqrt{L_{\tau(t)}^{n'}(v)} \in [\sqrt{t} + a_n(t) - k - 1, \sqrt{t} + a_n(t) - k] \right\}.$$

*Since the law of the simple random walk on  $T_{\leq n'}$  watched only on  $T_{\leq n}$  is the same as that of the simple random walk on  $T_{\leq n}$ , we have  $|\Gamma_k^{n',n}(t)| = |\Gamma_k^n(t)|$  in law.*

In the proof of Proposition 4.1, we will use the following repeatedly.

**Lemma 4.4** *(i) There exist  $c_1 > 0$  and  $t_0 > 0$  such that for all  $n \in \mathbb{N}$ ,  $t \geq t_0$ ,  $k \leq -1$ , and  $\lambda \in \mathbb{R}$  with  $k + \lambda \geq 0$ ,*

$$P_\rho \left( |\Gamma_k^n(t)| \geq e^{2\sqrt{\log b} (k+\lambda)} \right)$$

$$\leq c_1(\lambda + 1)e^{-2\sqrt{\log b}} \left( \lambda - \frac{3}{4\sqrt{\log b}} \log(\lceil (k+\lambda)^2 \rceil \vee 1) - \frac{1}{4\sqrt{\log b}} \log \left( \frac{\sqrt{t} + a_n(t) - k - 1 + \lceil (k+\lambda)^2 \rceil \vee 1}{\sqrt{t} + a_n(t) - k - 1} \right) \right). \quad (4.1)$$

(ii) There exist  $c_2 > 0$  and  $t_0 > 0$  such that for all  $n \in \mathbb{N}$ ,  $t \geq t_0$ ,  $\lambda > 0$ , and  $k \geq 0$  with  $\sqrt{t} + a_n(t) - k - 1 \geq c_2$ , (4.1) holds.

*Proof.* (i) Fix  $n \in \mathbb{N}$ ,  $t > 0$ ,  $k \leq -1$ , and  $\lambda \in \mathbb{R}$  with  $k + \lambda \geq 0$ . Set  $r := \lceil (k + \lambda)^2 \rceil \vee 1$  and  $y := \lambda - \frac{3}{4\sqrt{\log b}} \log r - \frac{1}{4\sqrt{\log b}} \log \left( \frac{\sqrt{t} + a_n(t) - k - 1 + r}{\sqrt{t} + a_n(t) - k - 1} \right)$ . If  $y < 0$ , then it is clear that (4.1) holds because  $P_\rho \left( |\Gamma_k^n(t)| \geq e^{2\sqrt{\log b}} (k + \lambda) \right) \leq 1 \leq (\lambda + 1)e^{-2\sqrt{\log b} y}$ . So, we may assume that  $y \geq 0$ . Fix any  $K > 0$ . We have

$$\begin{aligned} & P_\rho \left( \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}}(v) \geq \sqrt{t} + a_{n+r}(t) + y \right) \\ & \geq P_\rho \left( |\Gamma_k^{n+r,n}(t)| \geq K, \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}}(v) \geq \sqrt{t} + a_{n+r}(t) + y \right) \\ & \geq P_\rho \left( |\Gamma_k^{n+r,n}(t)| \geq K \right) - P_\rho \left( |\Gamma_k^{n+r,n}(t)| \geq K, \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}}(v) < \sqrt{t} + a_{n+r}(t) + y \right). \end{aligned} \quad (4.2)$$

We estimate the second term on the right-hand side of (4.2). By Lemma 2.1, we have

$$\begin{aligned} & P_\rho \left( |\Gamma_k^{n+r,n}(t)| \geq K, \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}}(v) < \sqrt{t} + a_{n+r}(t) + y \right) \\ & \leq \sum_{\substack{S \subset T_n, \\ |S| \geq K}} P_\rho \left( \Gamma_k^{n+r,n}(t) = S, \max_{v \in T_r^u} \sqrt{L_{\tau(t)}^{n+r}}(v) < \sqrt{t} + a_{n+r}(t) + y, \text{ for each } u \in S \right) \\ & = \sum_{\substack{S \subset T_n, \\ |S| \geq K}} E_\rho \left[ 1_{\{\Gamma_k^{n+r,n}(t) = S\}} \prod_{u \in S} \tilde{P}_u \left( \max_{v \in T_r^u} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t)}^{n+r})(u)} < \sqrt{t} + a_{n+r}(t) + y \right) \right], \end{aligned} \quad (4.3)$$

where for each  $u \in T_n$ ,  $\{\tilde{L}_s^\downarrow(x) : (s, x) \in [0, \infty) \times \tilde{T}_{\leq r}^u\}$  is a local time of a Brownian motion on  $\tilde{T}_{\leq r}^u$  and  $\tilde{\tau}^\downarrow(q) := \inf\{s \geq 0 : \tilde{L}_s^\downarrow(u) > q\}$ . We omit the subscript  $u$  in  $\tilde{L}_s^\downarrow(x)$  and  $\tilde{\tau}^\downarrow$  to simplify the notation.

We estimate each probability on the right-hand side of (4.3). Fix  $S \subset T_n$  with  $|S| \geq K$  and  $u \in S$ . Note that under the event that  $\Gamma_k^{n+r,n}(t) = S$ , we have

$$\sqrt{t} + a_{n+r}(t) + y \leq \sqrt{L_{\tau(t)}^{n+r}}(u) + a_r(L_{\tau(t)}^{n+r}(u)) + k + \lambda + 1,$$

$\sqrt{L_{\tau(t)}^{n+r}}(u) \geq \sqrt{t} + a_n(t) - k - 1 \geq \sqrt{t_0}$  if  $t \geq t_0$ , and  $k + \lambda + 1 \leq 2\sqrt{r}$  by the definition of  $r$ . By these, we can apply Proposition A.1 for  $t \geq t_0$ , where  $t_0$  is sufficiently large, and have

$$\tilde{P}_u \left( \max_{v \in T_r^u} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t)}^{n+r})(u)} < \sqrt{t} + a_{n+r}(t) + y \right)$$

$$\begin{aligned}
&\leq 1 - \tilde{P}_u \left( \max_{v \in T_r^u} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^{n+r}(u)}(v) \geq \sqrt{L_{\tau(t)}^{n+r}(u)} + a_r(L_{\tau(t)}^{n+r}(u)) + k + \lambda + 1 \right) \\
&\leq 1 - c_1 e^{-2\sqrt{\log b}(k+\lambda+1)}.
\end{aligned} \tag{4.4}$$

By (4.2)-(4.4), we have

$$\begin{aligned}
&P_\rho \left( \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}(v)} \geq \sqrt{t} + a_{n+r}(t) + y \right) \\
&\geq \left( 1 - \exp \left\{ -c_1 K e^{-2\sqrt{\log b}(k+\lambda+1)} \right\} \right) P_\rho \left( |\Gamma_k^{n+r,n}(t)| \geq K \right).
\end{aligned} \tag{4.5}$$

By Remark 4.3, (4.5) with  $K := e^{2\sqrt{\log b}(k+\lambda)}$ , and Proposition 3.1(i), we have (4.1).

(ii) The proof of (ii) is almost the same as that of (i), so we omit the detail.  $\square$

For the rest of this section, we focus on proving the following.

**Lemma 4.5** Fix  $0 < \underline{c} < \bar{c} < \frac{3}{4\sqrt{\log b}}$ . There exist  $c_1 > 0$ ,  $n_0, s_0 \in \mathbb{N}$ , and  $t_0 > 0$  such that for all  $n \geq n_0$ ,  $t \geq t_0$ , and  $s_0 \leq s \leq n - s_0$ ,

$$\begin{aligned}
&P_\rho \left[ \text{there exist } v, u \in T_n \text{ with } |v \wedge u| = s \text{ such that} \right. \\
&\quad \left. \sqrt{L_{\tau(t)}^n(v)}, \sqrt{L_{\tau(t)}^n(u)} \in [\sqrt{t} + a_n(t) - (\bar{c} - \underline{c}) \log(s \wedge (n-s)), \infty) \right] \\
&\leq c_1 (\log(s \wedge (n-s)))^8 (s \wedge (n-s))^{-3+4\bar{c}\sqrt{\log b}-2\underline{c}\sqrt{\log b}} \\
&\quad + c_1 (\log(s \wedge (n-s)))^6 (s \wedge (n-s))^{-2\underline{c}\sqrt{\log b}}.
\end{aligned} \tag{4.6}$$

Before we prove this, let us show that Lemma 4.5 implies Proposition 4.1.

*Proof of Proposition 4.1 via Lemma 4.5.* Fix any  $n \geq n_0$ ,  $t \geq t_0$ , and  $r_0 \leq r \leq n/4$ , where we take  $n_0, r_0 \in \mathbb{N}$  and  $t_0 > 0$  sufficiently large. By Lemma 4.5 with  $\underline{c} = \frac{5}{8\sqrt{\log b}}$  and  $\bar{c} = \frac{11}{16\sqrt{\log b}}$ , we have

$$\begin{aligned}
&P_\rho \left( \text{there exist } v, u \in T_n \text{ with } r \leq |v \wedge u| \leq n - r \text{ such that} \right. \\
&\quad \left. \sqrt{L_{\tau(t)}^n(v)}, \sqrt{L_{\tau(t)}^n(u)} \in [\sqrt{t} + a_n(t) - (\bar{c} - \underline{c}) \log r, \infty) \right) \\
&\leq c_1 \sum_{s=r}^{n-r} \left\{ (\log(s \wedge (n-s)))^8 (s \wedge (n-s))^{-3/2} + (\log(s \wedge (n-s)))^6 (s \wedge (n-s))^{-5/4} \right\} \\
&\leq c_2 \sum_{s=r}^{\infty} s^{-9/8} \leq c_3 r^{-1/8}. \quad \square
\end{aligned}$$

*Proof of Lemma 4.5.* Fix any  $n \geq n_0$ ,  $t \geq t_0$ ,  $s_0 \leq s \leq n - s_0$ , where we take  $n_0, s_0 \in \mathbb{N}$

and  $t_0 > 0$  sufficiently large. Set  $z := \underline{c} \log(s \wedge (n-s))$ . For  $k \in \mathbb{Z}$ , we set  $j^*(k, z) := \lceil \max\{|k|, z\} \rceil$ . Fix  $t_* > 0$ . Let  $P_s^n(t)$  be the probability on the left-hand side of (4.6). We have

$$\begin{aligned}
P_s^n(t) &\leq P_\rho \left[ \text{there exists } k \in \mathbb{Z} \text{ with } \sqrt{t} + a_s(t) - k > 0 \text{ and } j \geq -k \text{ such that} \right. \\
&\quad \left. |\Gamma_k^{n,s}(t)| \in \left[ e^{2\sqrt{\log b}(k+j)}, e^{2\sqrt{\log b}(k+j+1)} \right), \text{ and} \right. \\
&\quad \text{there exist } w \in \Gamma_k^{n,s}(t) \text{ and } w_1, w_2 \in T_1^w \text{ with } w_1 \neq w_2 \text{ such that} \\
&\quad \left. \max_{v \in T_{n-s-1}^{w_i}} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z, \text{ for all } i \in \{1, 2\} \right] \\
&\leq \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} \sum_{j=-k}^{j^*(k,z)} P_\rho \left[ |\Gamma_k^{n,s}(t)| \in \left[ e^{2\sqrt{\log b}(k+j)}, e^{2\sqrt{\log b}(k+j+1)} \right), \text{ and} \right. \\
&\quad \left. \text{there exist } w \in \Gamma_k^{n,s}(t) \text{ and } w_1, w_2 \in T_1^w \text{ with } w_1 \neq w_2 \text{ such that} \right. \\
&\quad \left. \max_{v \in T_{n-s-1}^{w_i}} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z, \text{ for all } i \in \{1, 2\} \right] \\
&\quad + \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} P_\rho \left( |\Gamma_k^{n,s}(t)| \geq e^{2\sqrt{\log b}(k+j^*(k,z))} \right) \\
&\quad + \sum_{\substack{k \in \mathbb{Z}, \\ 0 < \sqrt{t} + a_s(t) - k \leq t_* + 1}} P_\rho \left[ \text{there exist } w \in T_s \text{ with } \sqrt{L_{\tau(t)}^n(w)} \leq t_* + 1, \right. \\
&\quad \left. \text{and } w_1, w_2 \in T_1^w \text{ with } w_1 \neq w_2 \text{ such that} \right. \\
&\quad \left. \max_{v \in T_{n-s-1}^{w_i}} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z, \text{ for all } i \in \{1, 2\} \right] \\
&=: J_1 + J_2 + J_3. \tag{4.7}
\end{aligned}$$

For each  $w \in T_s$ , let  $\tilde{L}^\downarrow$  be a local time of a Brownian motion on  $\tilde{T}_{\leq n-s}^w$  and set  $\tilde{\tau}^\downarrow(q) := \inf\{p \geq 0 : \tilde{L}_p^\downarrow(w) > q\}$ . We omit the subscript  $w$  in  $\tilde{L}^\downarrow$  and  $\tilde{\tau}^\downarrow(q)$ . We first estimate  $J_1$ .

$$\begin{aligned}
J_1 &\leq \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} \sum_{j=-k}^{j^*(k,z)} \sum_{\substack{S \subset T_s, \\ |S| \in [e^{2\sqrt{\log b}(k+j)}, e^{2\sqrt{\log b}(k+j+1)})}} \sum_{\substack{w \in S, w_1, w_2 \in T_1^w, \\ w_1 \neq w_2}} P_\rho \left[ \Gamma_k^{n,s}(t) = S, \right. \\
&\quad \left. \max_{v \in T_{n-s-1}^{w_i}} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z, \text{ for all } i \in \{1, 2\} \right] \\
&= \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} \sum_{j=-k}^{j^*(k,z)} \sum_{\substack{S \subset T_s, \\ |S| \in [e^{2\sqrt{\log b}(k+j)}, e^{2\sqrt{\log b}(k+j+1)})}} \sum_{\substack{w \in S, w_1, w_2 \in T_1^w, \\ w_1 \neq w_2}} E_\rho \left[ 1_{\{\Gamma_k^{n,s}(t) = S\}} \right]
\end{aligned}$$

$$\begin{aligned}
& \prod_{i \in \{1, 2\}} \tilde{P}_w \left( \max_{v \in T_{n-s-1}^{w_i}} \sqrt{\tilde{L}_{\tilde{\tau}(L_{\tau(t)}^n(w))}^\downarrow}(v) \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z \right) \\
& \leq c_1 \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} \sum_{j=-k}^{j^*(k,z)} e^{2\sqrt{\log b}(k+j)} P_\rho \left( |\Gamma_k^{n,s}(t)| \geq e^{2\sqrt{\log b}(k+j)} \right) \\
& \quad \cdot P_\rho \left( \max_{v \in T_{n-s}} \sqrt{L_{\tau((\sqrt{t} + a_s(t) - k)^2)}^{n-s}}(v) \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z \right)^2, \quad (4.8)
\end{aligned}$$

where we have used Lemma 2.1 similarly to the argument in (3.35) in the second equality, and the symmetry of the  $b$ -ary tree and (2.3) in the last inequality. Note that for  $k \in \mathbb{Z}$  with  $k + z \geq 0$ , we have

$$\begin{aligned}
\sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z & \geq (\sqrt{t} + a_s(t) - k) + a_{n-s}((\sqrt{t} + a_s(t) - k)^2) \\
& \quad + \left( \frac{3}{4\sqrt{\log b}} - \bar{c} \right) \log(s \wedge (n-s)) + k + z - c_2.
\end{aligned}$$

By this and Proposition 3.1(i) together with (4.8), Remark 4.3, and Lemma 4.4, we have for sufficiently large  $t_*$

$$\begin{aligned}
J_1 & \leq c_3 (\log(s \wedge (n-s)))^8 (s \wedge (n-s))^{-3+4\bar{c}\sqrt{\log b}-2\underline{c}\sqrt{\log b}} \\
& \quad + c_3 (\log(s \wedge (n-s))) (s \wedge (n-s))^{-2\underline{c}\sqrt{\log b}}. \quad (4.9)
\end{aligned}$$

Next, we estimate  $J_2$ . By Lemma 4.4 and Remark 4.3, we have

$$J_2 \leq c_4 (\log(s \wedge (n-s)))^6 (s \wedge (n-s))^{-2\underline{c}\sqrt{\log b}}. \quad (4.10)$$

Next, we estimate  $J_3$ . By using Lemma 2.1 in a way similar to the argument in (4.8), we have

$$\begin{aligned}
J_3 & \leq c_5 \cdot \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k \in (0, t_* + 1]}} b^s \\
& \quad P_\rho \left( \max_{v \in T_{n-s}} \sqrt{L_{\tau((t_*+1)^2)}^{n-s}}(v) \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z \right)^2. \quad (4.11)
\end{aligned}$$

By Proposition 3.1(i) and (4.11), we have

$$J_3 \leq c_6 s^6 b^{-3s} (s \wedge (n-s))^{4(\bar{c}-\underline{c})\sqrt{\log b}}. \quad (4.12)$$

Thus, by (4.7), (4.9), (4.10), and (4.12), we have (4.6).  $\square$

## 5 Limiting tail of the maximum of local times

The aim of this section is to prove the following.

**Proposition 5.1** Fix positive sequences  $(y_j^+)_{j \geq 1}, (y_j^-)_{j \geq 1}$  with  $y_j^- \leq y_j^+$  for each  $j \geq 1$  and  $\lim_{j \rightarrow \infty} y_j^- = \infty$ . For each  $j \geq 1$ , fix sequences  $(t_n^+(j))_{n \geq 1}, (t_n^-(j))_{n \geq 1}$  with  $t_n^-(j) \leq t_n^+(j)$  for each  $n \geq 1$  and  $\lim_{n \rightarrow \infty} \sqrt{t_n^+(j)}/n = \lim_{n \rightarrow \infty} \sqrt{t_n^-(j)}/n = \theta \in [0, \infty]$ . There exists  $c_* > 0$  such that the following holds: for all  $\varepsilon > 0$ , there exists  $j_0 \in \mathbb{N}$  such that for each  $j \geq j_0$ , there exists  $n_0(j) \in \mathbb{N}$  such that for all  $n \geq n_0(j)$ ,

$$\frac{P_\rho \left( \max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y_j \right)}{y_j e^{-2\sqrt{\log b} y_j}} \in \left[ \frac{4}{\sqrt{\pi}} \beta_* \gamma_* - \varepsilon, \frac{4}{\sqrt{\pi}} \beta_* \gamma_* + \varepsilon \right], \quad (5.1)$$

uniformly in  $y_j$  and  $t$  satisfying that

$$y_j^- \leq y_j \leq y_j^+, \quad t \geq c_* n \log n \text{ and } t_n^-(j) \leq t \leq t_n^+(j), \quad (5.2)$$

where  $\beta_*$  and  $\gamma_*$  are the constants defined in (1.8) and (1.9).

We begin with some notation. Fix  $\delta \in (0, 1)$  and  $\kappa \in \left( \frac{1}{2\sqrt{\log b}}, \infty \right)$ . For all  $t > 0, y > 0$ ,  $\ell(y) \in \mathbb{N}$ ,  $n > \ell(y)$ , and  $v \in T_{n-\ell(y)}$ , set

$$F_{v,y,\ell(y)}^n(t) := \left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + y, \text{ for all } 0 \leq s \leq n - \ell(y), \right. \\ \left. \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tau(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right\},$$

$$\tilde{F}_{v,y,\ell(y)}^n(t) := \left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + y + y^{1/20} + \kappa(\log(s \wedge (n - \ell(y) - s)))_+, \right. \\ \left. \text{for all } 0 \leq s \leq n - \ell(y), \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tau(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right\},$$

$$\Lambda_{y,\ell(y)}^n(t) := \sum_{v \in T_{n-\ell(y)}} 1_{F_{v,y,\ell(y)}^n(t)}, \quad \tilde{\Lambda}_{y,\ell(y)}^n(t) := \sum_{v \in T_{n-\ell(y)}} 1_{\tilde{F}_{v,y,\ell(y)}^n(t)}.$$

In Lemma 5.4, we show that  $P_\rho(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y) / \tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)]$  is close to 1. To obtain an upper bound of this, we need the following:

**Lemma 5.2** There exist  $c_1, c_2 \in (0, \infty)$ ,  $y_0 > 0$ , and  $\{\delta_{y'} : y' > 0\}$  with  $\lim_{y' \rightarrow \infty} \delta_{y'} = 0$  such that the following holds: for all  $y \geq y_0$  and  $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$ , there exists  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$  such that for all  $n \geq n_0$  and  $t \geq c_1 n \log n$ ,

$$\frac{\tilde{E}_\rho \left[ \Lambda_{y,\ell(y)}^n(t) \right]}{\tilde{E}_\rho \left[ \tilde{\Lambda}_{y,\ell(y)}^n(t) \right]}$$

$$\geq (1 - \delta_y) \left( 1 - c_2(\ell(y))^{-3/2} (\log \ell(y))^{60} - \delta_y y^{-1} (\ell(y))^{-3/2} n^3 (n + \sqrt{t}) e^{-\frac{(1-\delta)^2}{n-\ell(y)} t} \right). \quad (5.3)$$

*Proof.* Fix any  $y \geq y_0$  and  $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$ , where we take  $y_0 > 0$  large enough. Throughout the proof, given  $n \in \mathbb{N}$ , we assume that  $t \geq c_* n \log n$ , where  $c_*$  is a sufficiently large positive constant. Fix  $v \in T_{n-\ell(y)}$ . Let  $\tilde{L}^\downarrow$  be a local time of a Brownian motion on  $\tilde{T}_{\leq \ell(y)}^v$  and set  $\tilde{\tau}^\downarrow(s) := \inf\{r \geq 0 : \tilde{L}_r^\downarrow(v) > s\}$ . Recall  $\mu_{n-\ell(y),y}$ ,  $\mu_{n-\ell(y),y}^*$ ,  $\delta_y$  in Lemma 2.6. We have

$$\begin{aligned} & \tilde{P}_\rho \left( \tilde{F}_{v,y,\ell(y)}^n(t) \right) - \tilde{P}_\rho \left( F_{v,y,\ell(y)}^n(t) \right) \\ &= \tilde{E}_\rho \left[ \left( 1 \left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + y^{1/20} + \kappa(\log(s \wedge (n-\ell(y)-s)))_+, \text{ for all } 0 \leq s \leq n-\ell(y) \right\} \right. \right. \\ & \quad \left. \left. - 1 \left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + y, \text{ for all } 0 \leq s \leq n-\ell(y) \right\} \right) \right. \\ & \quad \left. \cdot \tilde{P}_v \left( \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(t)}^\downarrow(L_{\tilde{\tau}^\downarrow(t)}^n(v))(u)} > \sqrt{t} + a_n(t) + y \right) \right] \\ &\leq \int_{[0, y+y^{1/20}] \cup [-(1-\delta)\sqrt{t} - \frac{a_n(t)}{n}(n-\ell(y)), -\ell(y)]} \mu_{n-\ell(y),y}^*(x) \psi(x) dx \\ & \quad + \int_{-\ell(y)}^0 \delta_y \mu_{n-\ell(y),y}(x) \psi(x) dx \\ &=: J_1 + J_2, \end{aligned} \quad (5.4)$$

where

$$\begin{aligned} \psi(x) &:= \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + x}} \cdot e^{-\frac{2a_n(t)}{n} x - \frac{(a_n(t))^2}{n^2} (n-\ell(y))} \\ & \quad \cdot \tilde{P}_v \left( \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(t)}^\downarrow((\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + x)^2)}(u) > \sqrt{t} + a_n(t) + y \right), \end{aligned}$$

and we have used Lemma 2.1 in the first equality, Lemma 2.5, (2.7), (3.14), (3.15), and Lemma 2.6 in the second inequality. By Proposition 3.1(i) and the assumption  $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$ , taking  $n_0(y, \ell(y)) \in \mathbb{N}$  large enough, we have for all  $n \geq n_0(y, \ell(y))$

$$J_1 \leq c_1 b^{-(n-\ell(y))} y e^{-2\sqrt{\log b} y} (\ell(y))^{-3/2} (\log \ell(y))^{60}. \quad (5.5)$$

Next, we estimate  $J_2$ . We have

$$J_2 \leq \delta_y E_0^B \left[ 1 \left\{ -(1-\delta)\sqrt{t} \leq B_s/\sqrt{2} \leq \frac{a_n(t)}{n} s + y, \text{ for all } 0 \leq s \leq n-\ell(y) \right\} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + B_{n-\ell(y)}/\sqrt{2}}} \right]$$

$$\begin{aligned}
& \cdot \tilde{P}_v \left( \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow((\sqrt{t} + B_{n-\ell(y)})/\sqrt{2})^2}(u) > \sqrt{t} + a_n(t) + y \right) \Big] \\
& + \delta_y E_0^B \left[ 1_{\left\{ -(1-\delta)\sqrt{t} \leq B_{n-\ell(y)}/\sqrt{2} \leq \frac{a_n(t)}{n}(n-\ell(y)), \min_{0 \leq s \leq n-\ell(y)} B_s/\sqrt{2} < -(1-\delta)\sqrt{t} \right\}} \right. \\
& \quad \left. \sqrt{\frac{\sqrt{t}}{\sqrt{t} + B_{n-\ell(y)}/\sqrt{2}}} \cdot \tilde{P}_v \left( \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow((\sqrt{t} + B_{n-\ell(y)})/\sqrt{2})^2}(u) > \sqrt{t} + a_n(t) + y \right) \right] \\
& =: J_{2,1} + J_{2,2}. \tag{5.6}
\end{aligned}$$

By Lemma 2.5 and (2.7), we have for each  $n \geq 1$

$$J_{2,1} \leq c_2 \delta_y \tilde{P}_\rho(F_{v,y,\ell(y)}^n(t)), \tag{5.7}$$

where we have used that under the event that  $\sqrt{X_s/2} \geq \delta\sqrt{t}$  for all  $0 \leq s \leq n-\ell(y)$ , we have  $\exp\left(\frac{3}{8} \int_0^{n-\ell(y)} \frac{ds}{X_s}\right) \leq c_2$  under the assumption  $t \geq c_* n \log n$ .

By (3.25), the symmetry of  $B$  and Proposition 3.1(i), taking  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$  large enough, we have for all  $n \geq n_0$

$$J_{2,2} \leq c_3 \delta_y b^{-(n-\ell(y))} e^{-2\sqrt{\log b} y} (\ell(y))^{-3/2} n^3 (n + \sqrt{t}) e^{-\frac{(1-\delta)^2 t}{n-\ell(y)}}. \tag{5.8}$$

Thus, by (5.4)-(5.8), we have

$$\begin{aligned}
(1 + c_2 \delta_y) \tilde{E}_\rho(\Lambda_{y,\ell(y)}^n(t)) & \geq \tilde{E}_\rho(\tilde{\Lambda}_{y,\ell(y)}^n(t)) - c_1 y e^{-2\sqrt{\log b} y} (\ell(y))^{-3/2} (\log \ell(y))^{60} \\
& \quad - c_3 \delta_y e^{-2\sqrt{\log b} y} (\ell(y))^{-3/2} n^3 (n + \sqrt{t}) e^{-\frac{(1-\delta)^2 t}{n-\ell(y)}}. \tag{5.9}
\end{aligned}$$

For the rest of the proof, we obtain a lower bound of  $\tilde{E}_\rho(\tilde{\Lambda}_{y,\ell(y)}^n(t))$ . Recall (3.3). We have

$$\begin{aligned}
& \tilde{E}_\rho(\tilde{\Lambda}_{y,\ell(y)}^n(t)) \\
& \geq \tilde{P}_\rho \left[ \left\{ \text{there exists } v \in T_{n-\ell(y)} \text{ such that } \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \geq \delta\sqrt{t}, \text{ for all } 0 \leq s \leq n-\ell(y), \right. \right. \\
& \quad \left. \left. \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right\} \cap \left( G_{y+y^{1/20}-1}^{n-\ell(y)}(t) \right)^c \right] \\
& \geq P_\rho \left( \max_{u \in T_n} \sqrt{L_{\tau(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right) - \tilde{P}_\rho \left( G_{y+y^{1/20}-1}^{n-\ell(y)}(t) \right) \\
& \quad - \tilde{P}_\rho \left[ \text{there exists } v \in T_{n-\ell(y)} \text{ such that } \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \delta\sqrt{t}, \text{ for some } 0 \leq s \leq n-\ell(y), \right. \\
& \quad \left. \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right]. \tag{5.10}
\end{aligned}$$



We estimate the third term on the right-hand side of (5.10). Fix  $v^* \in T_{n-\ell(y)}$ . By Lemma 2.1, 2.5, and (2.7), taking  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$  large enough, we have for all  $n \geq n_0$

$$\begin{aligned}
& \tilde{P}_\rho \left[ \text{there exists } v \in T_{n-\ell(y)} \text{ such that} \right. \\
& \quad \left. \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \delta \sqrt{t}, \text{ for some } 0 \leq s \leq n - \ell(y), \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right] \\
& \leq b^{n-\ell(y)} \tilde{E}_\rho \left[ 1_{\left\{ \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s^*)} < \delta \sqrt{t}, \text{ for some } 0 \leq s \leq n - \ell(y), \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v^*)} > 0 \right\}} \right. \\
& \quad \left. \cdot \tilde{P}_{v^*} \left( \max_{u \in T_{\ell(y)}^{v^*}} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^\downarrow(\tilde{L}_{\tilde{\tau}(t)}^n(v^*))}(u)} > \sqrt{t} + a_n(t) + y \right) \right] \\
& \leq b^{n-\ell(y)} E_0^B \left[ 1_{\left\{ B_s/\sqrt{2} < -(1-\delta)\sqrt{t}, \text{ for some } 0 \leq s \leq n - \ell(y), B_{n-\ell(y)}/\sqrt{2} > -\sqrt{t} \right\}} \sqrt{\frac{\sqrt{2t}}{\sqrt{2t} + B_{n-\ell(y)}}} \right. \\
& \quad \left. \cdot \tilde{P}_{v^*} \left( \max_{u \in T_{\ell(y)}^{v^*}} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^\downarrow((\sqrt{t} + B_{n-\ell(y)}/\sqrt{2})^2)}(u)} > \sqrt{t} + a_n(t) + y \right) \right] \\
& \leq c_4(\ell(y))^{-3/2} (n + \sqrt{t})^3 e^{-2(1-\delta)\sqrt{\log b} \sqrt{t}} e^{-2\sqrt{\log b} y}, \tag{5.11}
\end{aligned}$$

where we have used Proposition 3.1(i) and (3.25). By (5.10) and (5.11) together with Proposition 3.1(ii) and Lemma 3.2, taking  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$  large enough, we have for all  $n \geq n_0$

$$\tilde{E}_\rho(\tilde{\Lambda}_{y, \ell(y)}^n(t)) \geq c_5 y e^{-2\sqrt{\log b} y}. \tag{5.12}$$

By (5.9) and (5.12), we have (5.3).  $\square$

To obtain a lower bound of  $P_\rho(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y) / \tilde{E}_\rho[\Lambda_{y, \ell(y)}^n(t)]$ , we need the following:

**Lemma 5.3** *There exist  $c_1, c_2, c_3 \in (0, \infty)$ ,  $y_0 > 0$  such that the following holds: for all  $y \geq y_0$  and  $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$ , there exists  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$  such that for all  $n \geq n_0$  and  $t \geq c_1 n \log n$ ,*

$$\frac{\tilde{E}_\rho \left[ \left( \Lambda_{y, \ell(y)}^n(t) \right)^2 \right]}{\tilde{E}_\rho \left[ \Lambda_{y, \ell(y)}^n(t) \right]} \leq 1 + c_2 y^{-1/2} + c_3 y e^{-2\sqrt{\log b} y}. \tag{5.13}$$

*Proof.* Fix any  $y \geq y_0$  and  $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$ , where we take  $y_0 > 0$  large enough. Throughout the proof, given  $n \in \mathbb{N}$ , we assume  $t \geq c_* n \log n$  for some sufficiently large

$c_* > 0$ . We have

$$\tilde{E}_\rho \left[ \left( \Lambda_{y, \ell(y)}^n(t) \right)^2 \right] = \tilde{E}_\rho [\Lambda_{y, \ell(y)}^n(t)] + \sum_{k=1}^{n-\ell(y)} \sum_{\substack{v, w \in T_{n-\ell(y)}, \\ |v \wedge w| = n-\ell(y)-k}} \tilde{P}_\rho \left( F_{v, y, \ell(y)}^n(t) \cap F_{w, y, \ell(y)}^n(t) \right). \quad (5.14)$$

Fix  $1 \leq k \leq n - \ell(y) - 1$  and  $v, w \in T_{n-\ell(y)}$  with  $|v \wedge w| = n - \ell(y) - k$ . Let  $\tilde{L}^\downarrow$  be a local time of a Brownian motion on  $\tilde{T}_{\leq \ell(y)+k}^{v \wedge w}$  and set  $\tilde{\tau}^\downarrow(s) := \inf\{r \geq 0 : \tilde{L}_r^\downarrow(v \wedge w) > s\}$ . We have

$$\begin{aligned} & \tilde{P}_\rho \left( F_{v, y, \ell(y)}^n(t) \cap F_{w, y, \ell(y)}^n(t) \right) \\ &= \tilde{E}_\rho \left[ 1_{\left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(t)}^\downarrow(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + y, \text{ for all } 0 \leq s \leq n - \ell(y) - k \right\}} \prod_{x \in \{v, w\}} P^x \right], \end{aligned} \quad (5.15)$$

where for each  $x \in \{v, w\}$ ,

$$\begin{aligned} P^x &:= \tilde{P}_{v \wedge w} \left( \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(t)}^\downarrow(v \wedge w)}(x_s) \leq \sqrt{t} + \frac{a_n(t)}{n} s + y, \right. \\ &\quad \left. \text{for all } n - \ell(y) - k \leq s \leq n - \ell(y), \max_{u \in T_{\ell(y)}^x} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(t)}^\downarrow(v \wedge w)}(u) > \sqrt{t} + a_n(t) + y \right), \end{aligned}$$

and we have used Lemma 2.1 in a way similar to the argument in (3.35).

Fix  $0 \leq i \leq \lfloor (1 - \delta) \sqrt{t} + \frac{a_n(t)}{n} (n - \ell(y) - k) + y \rfloor$ . Assume that

$$\sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(t)}^\downarrow(v \wedge w)} \in \sqrt{t} + \frac{a_n(t)}{n} (n - \ell(y) - k) + y - i + (-1, 0]. \quad (5.16)$$

Under the assumption (5.16), we estimate the probabilities in (5.15). Let  $\tilde{L}^{\downarrow\downarrow}$  be a local time of a Brownian motion on  $\tilde{T}_{\leq \ell(y)}^v$  and set  $\tilde{\tau}^{\downarrow\downarrow}(s) := \inf\{r \geq 0 : \tilde{L}_r^{\downarrow\downarrow}(v) > s\}$ . Using Lemma 2.1, Lemma 2.5, (2.7), (5.16), and the density (2.12) in a way similar to the argument in (5.4), we have

$$\begin{aligned} P^v &\leq \int_0^{(1-\delta)\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y))+y+1} \frac{1}{\sqrt{\pi k}} \left( e^{-\frac{(i+1-z)^2}{k}} - e^{-\frac{(i+1+z)^2}{k}} \right) \\ &\quad e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}(i+1) - \frac{(a_n(t))^2}{n^2}k} \sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y) - k) + y - i}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y - z}} \\ &\quad \cdot \tilde{P}_v \left( \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^{\downarrow\downarrow}(t)}^{\downarrow\downarrow}} \left( \left( \sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y + 1 - z \right)^2 \right) (u) > \sqrt{t} + a_n(t) + y \right) dz. \end{aligned} \quad (5.17)$$

We use the following estimate:  $e^{-\frac{(i+1-z)^2}{k}} - e^{-\frac{(i+1+z)^2}{k}} \leq 1$  for each  $1 \leq k \leq \lfloor y \rfloor$ , and  $e^{-\frac{(i+1-z)^2}{k}} - e^{-\frac{(i+1+z)^2}{k}} \leq \frac{4(i+1)z}{k}$  for each  $\lfloor y \rfloor \leq k \leq n - \ell(y) - 1$ . By this and Proposition

3.1(i), the right-hand side of (5.17) is bounded from above by

$$c_1 A b^{-k} e^{-\frac{2a_n(t)}{n}i} e^{\frac{3k \log n}{2n}} e^{\frac{k \log \left( \frac{\sqrt{t}+n}{\sqrt{t}} \right)}{2n}} \sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y) - k) + y - i}{\sqrt{t} + n}}, \quad (5.18)$$

where  $A := k^{-1/2}(\ell(y))^{-1/2}$  if  $1 \leq k \leq \lfloor y \rfloor$ , and  $A := k^{-3/2}(i+1)$  if  $\lfloor y \rfloor \leq k \leq n - \ell(y) - 1$ . Recall the events in the indicator function in (5.15) and in (5.16). We estimate the probability of the intersection of these events. Using Lemma 2.5, (2.7), and the density (2.12) similarly to the proof of Lemma 3.4 and taking  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$  large enough, we have for  $n \geq n_0$

$$\begin{aligned} & \tilde{P}_\rho \left( \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n}(v_s) \leq \sqrt{t} + \frac{a_n(t)}{n}s + y, \text{ for all } 0 \leq s \leq n - \ell(y) - k, \right. \\ & \quad \left. \sqrt{L_{\tau(t)}^n}(v \wedge w) \in \sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y) - k) + y - i + (-1, 0] \right) \\ & \leq \int_i^{i+1} \frac{1}{\sqrt{\pi(n - \ell(y) - k)}} \left( e^{-\frac{(y-z)^2}{n - \ell(y) - k}} - e^{-\frac{(y+z)^2}{n - \ell(y) - k}} \right) \\ & \quad e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}y - \frac{(a_n(t))^2}{n^2}(n - \ell(y) - k)} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y) - k) + y - z}} dz. \quad (5.19) \end{aligned}$$

We will use the following:  $e^{-\frac{(y-z)^2}{n - \ell(y) - k}} - e^{-\frac{(y+z)^2}{n - \ell(y) - k}} \leq \frac{4yz}{n - \ell(y) - k}$  for each  $1 \leq k \leq n - \ell(y) - \lfloor y \rfloor$ , and  $e^{-\frac{(y-z)^2}{n - \ell(y) - k}} - e^{-\frac{(y+z)^2}{n - \ell(y) - k}} \leq 1$  for each  $n - \ell(y) - \lfloor y \rfloor \leq k \leq n - \ell(y) - 1$ . By this, the right-hand side of (5.19) is bounded from above by

$$\begin{aligned} & c_2 A' b^{-(n - \ell(y) - k)} e^{\frac{2a_n(t)}{n}i} e^{-2\sqrt{\log b} y} \\ & \cdot e^{\frac{3(n - \ell(y) - k)}{2n} \log n} e^{\frac{(n - \ell(y) - k) \log \left( \frac{\sqrt{t}+n}{\sqrt{t}} \right)}{2n}} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y) - k) + y - i - 1}}, \quad (5.20) \end{aligned}$$

where  $A' := (n - \ell(y) - k)^{-3/2}(i+1)y$  if  $1 \leq k \leq n - \ell(y) - \lfloor y \rfloor$ , and  $A' := (n - \ell(y) - k)^{-1/2}$  if  $n - \ell(y) - \lfloor y \rfloor \leq k \leq n - \ell(y) - 1$ . By (5.15), (5.18), and (5.20), we have

$$\sum_{k=1}^{n - \ell(y) - 1} \sum_{\substack{v, w \in T_{n - \ell(y)}, \\ |v \wedge w| = n - \ell(y) - k}} \tilde{P}_\rho \left( F_{v, y, \ell(y)}^n(t) \cap F_{w, y, \ell(y)}^n(t) \right) \leq c_3 y^{1/2} e^{-2\sqrt{\log b} y}. \quad (5.21)$$

For the case  $k = n - \ell(y)$ , by the independence of excursions of a Brownian motion around  $\rho$ , we have

$$\sum_{\substack{v, w \in T_{n - \ell(y)}, \\ |v \wedge w| = 0}} \tilde{P}_\rho \left( F_{v, y, \ell(y)}^n(t) \cap F_{w, y, \ell(y)}^n(t) \right) = \sum_{\substack{v, w \in T_{n - \ell(y)}, \\ |v \wedge w| = 0}} \tilde{P}_\rho \left( F_{v, y, \ell(y)}^n(t) \right) \tilde{P}_\rho \left( F_{w, y, \ell(y)}^n(t) \right)$$

$$\leq \left( \tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)] \right)^2. \quad (5.22)$$

Thus, by (5.14), (5.21), and (5.22), we have

$$\tilde{E}_\rho \left[ \left( \Lambda_{y,\ell(y)}^n(t) \right)^2 \right] \leq \tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)] + c_3 y^{1/2} e^{-2\sqrt{\log b} y} + \left( \tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)] \right)^2. \quad (5.23)$$

We will obtain upper and lower bounds of  $\tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)]$ . By (5.9) and (5.12), taking  $n_0 = n_0(y, \ell(y))$  large enough, we have for all  $n \geq n_0$  (recall that we assumed  $t \geq c_* n \log n$  and  $c_*$  is sufficiently large positive constant)

$$\tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)] \geq c_4 y e^{-2\sqrt{\log b} y}. \quad (5.24)$$

By using Lemma 2.1,2.5, (2.7), the density (2.12), and Proposition 3.1(i) similarly to the argument in (5.17), we have

$$\tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)] \leq c_5 y e^{-2\sqrt{\log b} y}. \quad (5.25)$$

Thus, by (5.23)-(5.25), we have (5.13).  $\square$

Using Lemma 5.2 and 5.3, we prove the following:

**Lemma 5.4** *There exist  $c_1, c_2, c_3 \in (0, \infty)$ ,  $y_0 > 0$ , and  $\{\delta_{y'} : y' > 0\}$  with  $\lim_{y' \rightarrow \infty} \delta_{y'} = 0$  such that the following holds: for all  $y \geq y_0$  and  $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$ , there exists  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$  such that for all  $n \geq n_0$  and  $t \geq c_1 n \log n$ ,*

$$\begin{aligned} & \frac{P_\rho \left( \max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y \right)}{\tilde{E}_\rho \left[ \Lambda_{y,\ell(y)}^n(t) \right]} \\ & \leq (1 + \delta_y) \left( 1 + c_2 e^{-2\sqrt{\log b} y^{1/20}} + c_2 (\ell(y))^{-3/2} (n + \sqrt{t})^3 e^{-2\sqrt{\log b} (1-\delta) \sqrt{t}} \right) \\ & \quad \cdot \left( 1 + c_2 (\ell(y))^{-3/2} (\log \ell(y))^{60} + c_2 \delta_y y^{-1} (\ell(y))^{-3/2} n^3 (n + \sqrt{t}) e^{-\frac{(1-\delta)^2}{n-\ell(y)} t} \right), \quad (5.26) \end{aligned}$$

$$\frac{P_\rho \left( \max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y \right)}{\tilde{E}_\rho \left[ \Lambda_{y,\ell(y)}^n(t) \right]} \geq 1 - c_3 y^{-1/2} - c_3 y e^{-2\sqrt{\log b} y}. \quad (5.27)$$

*Proof.* Fix any  $y \geq y_0$ ,  $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$ ,  $n \geq n_0$ , and  $t \geq c_* n \log n$ , where we take  $y_0 > 0$ ,  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ ,  $c_* > 0$  large enough. Recall (3.3). We first obtain the upper bound. We have

$$P_\rho \left( \max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y \right)$$

$$\begin{aligned}
&= P_\rho \left( \text{there exists } v \in T_{n-\ell(y)} \text{ such that } \max_{u \in T_{\ell(y)}^v} \sqrt{L_{\tau(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right) \\
&\leq \tilde{E}_\rho[\tilde{\Lambda}_{y,\ell(y)}^n(t)] + \tilde{P}_\rho \left( G_{y+y^{1/20}-1}^{n-\ell(y)}(t) \right) \\
&\quad + \tilde{P}_\rho \left( \text{there exists } v \in T_{n-\ell(y)} \text{ such that } \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tau(t)}^n(u)} > \sqrt{t} + a_n(t) + y, \right. \\
&\quad \left. \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} < \delta \sqrt{t}, \text{ for some } 0 \leq s \leq n - \ell(y) \right) \\
&\leq \left( 1 + c_1 e^{-2\sqrt{\log b} y^{1/20}} + c_1 (\ell(y))^{-3/2} (n + \sqrt{t})^3 e^{-2\sqrt{\log b} (1-\delta)\sqrt{t}} \right) \tilde{E}_\rho[\tilde{\Lambda}_{y,\ell(y)}^n(t)],
\end{aligned} \tag{5.28}$$

where we have used Lemma 3.2, (5.11), and (5.12) in the last inequality. By (5.28) and Lemma 5.2, we have (5.26). By Lemma 5.3, we have

$$\begin{aligned}
P_\rho \left( \max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y \right) &\geq \tilde{P}_\rho \left( \Lambda_{y,\ell(y)}^n(t) \geq 1 \right) \geq \frac{\left( \tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)] \right)^2}{\tilde{E}_\rho \left[ \left( \Lambda_{y,\ell(y)}^n(t) \right)^2 \right]} \\
&\geq \left( 1 - c_2 y^{-1/2} - c_2 y e^{-2\sqrt{\log b} y} \right) \tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)],
\end{aligned}$$

which proves (5.27).  $\square$

For each interval  $I \subset \mathbb{R}$ , set

$$\Lambda_{y,\ell(y),I}^n(t) := \sum_{v \in T_{n-\ell(y)}} 1_{F_{v,y,\ell(y)}^n(t) \cap \left\{ \sqrt{\tilde{L}_{\tau(t)}^n(v)} \in \sqrt{t} + \frac{a_n(t)}{n} (n - \ell(y)) + I \right\}}. \tag{5.29}$$

Set the interval

$$J_{\ell(y)} := \left( -\ell(y) + y, -(\ell(y))^{2/5} + y \right]. \tag{5.30}$$

Then the following holds:

**Lemma 5.5** *There exist  $c_1, c_2 \in (0, \infty)$  and  $y_0 > 0$  such that the following holds: for all  $y \geq y_0$  and  $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$ , there exists  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$  such that for all  $n \geq n_0$  and  $t \geq c_1 n \log n$ ,*

$$\frac{\tilde{E}_\rho \left[ \Lambda_{y,\ell(y),J_{\ell(y)}}^n(t) \right]}{\tilde{E}_\rho \left[ \Lambda_{y,\ell(y)}^n(t) \right]} \geq 1 - c_2 (\ell(y))^{-3/10}. \tag{5.31}$$

*Proof.* Fix any  $y \geq y_0$  and  $\ell(y) > e^{\frac{8\sqrt{\log b}}{3} y^{1/20}}$ , where we take  $y_0 > 0$  large enough. Throughout the proof, given  $n \in \mathbb{N}$ , we assume that  $t \geq c_* n \log n$  for some sufficiently large  $c_* > 0$ . Fix  $v \in T_{n-\ell(y)}$  and any interval  $I \subset \mathbb{R}$ . By using Lemma 2.1, 2.5, (2.7),

the density (2.12), and Proposition 3.1(i) similarly to the argument in (5.25), taking  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$  large enough, we have for all  $n \geq n_0$

$$\begin{aligned} & \tilde{P}_\rho \left( F_{v,y,\ell(y)}^n(t) \cap \left\{ \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v)} \in \sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + I \right\} \right) \\ & \leq c_1 b^{-(n-\ell(y))} (\ell(y))^{-3/2} y e^{-2\sqrt{\log b} y} \\ & \cdot \int_{(y-I) \cap [0, (1-\delta)\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y))+y]} z (\log \ell(y) + z) e^{-c_2 \frac{z^2}{\ell(y)}} \\ & \cdot \sqrt{\frac{\sqrt{t} + n}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y - z + \ell(y)}} dz. \end{aligned} \quad (5.32)$$

In the cases  $I = (-\infty, y - \ell(y)]$  and  $I = (y - (\ell(y))^{2/5}, \infty)$ , the right-hand side of (5.32) is bounded from above by

$$c_3 b^{-(n-\ell(y))} (\ell(y))^{-3/10} y e^{-2\sqrt{\log b} y}$$

for all  $n \geq n_0$ , where  $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$  large enough. By this and (5.24), we have

$$\begin{aligned} & \tilde{E}_\rho \left[ \Lambda_{y,\ell(y)}^n(t) \right] \\ & = \tilde{E}_\rho \left[ \Lambda_{y,\ell(y),J_{\ell(y)}}^n(t) \right] + \tilde{E}_\rho \left[ \Lambda_{y,\ell(y),(-\infty, y-\ell(y)]}^n(t) \right] + \tilde{E}_\rho \left[ \Lambda_{y,\ell(y),(y-(\ell(y))^{2/5}, \infty)}^n(t) \right] \\ & \leq \tilde{E}_\rho \left[ \Lambda_{y,\ell(y),J_{\ell(y)}}^n(t) \right] + c_4 (\ell(y))^{-3/10} \tilde{E}_\rho \left[ \Lambda_{y,\ell(y)}^n(t) \right], \end{aligned}$$

which implies (5.31).  $\square$

*Proof of Proposition 5.1.* Let  $(h_v)_{v \in T}$  be a BRW on  $T$  defined in Section 1. By Lemma A.5, one can show that the sequence

$$\left( \int_{\ell^{2/5}}^\ell z e^{2\sqrt{\log b} z} \mathbb{P} \left[ \max_{u \in T_\ell} h_u / \sqrt{2} > \sqrt{\log b} \ell + z \right] dz \right)_{\ell \geq 1} \quad (5.33)$$

is bounded from above and away from 0. Fix a nondecreasing sequence  $(\ell_0(y_k^+))_{k \geq 1}$  with  $\ell_0(y_k^+) > e^{\frac{8\sqrt{\log b}}{3}(y_k^+)^{1/20}}$  for each  $k \geq 1$ . By the boundedness of the sequence (5.33), there exists a subsequence  $(\ell_0(y_{k_j}^+))_{j \geq 1}$  of  $(\ell_0(y_k^+))_{k \geq 1}$  such that the limit

$$\tilde{\gamma}_* := \lim_{j \rightarrow \infty} \int_{(\ell_0(y_{k_j}^+))^{2/5}}^{\ell_0(y_{k_j}^+)} z e^{2\sqrt{\log b} z} \mathbb{P} \left[ \max_{u \in T_{\ell_0(y_{k_j}^+)}} h_u / \sqrt{2} > \sqrt{\log b} \ell_0(y_{k_j}^+) + z \right] dz \in (0, \infty) \quad (5.34)$$

exists. We set

$$\ell_j := \ell_0(y_{k_j}^+), \quad j \geq 1.$$

Note that by the definition of  $\ell_0(y_j^+)$ , for any  $y_j$  with  $y_j \leq y_j^+$ , we have

$$\ell_j \geq \ell_0(y_j^+) > e^{\frac{8\sqrt{\log b}}{3}(y_j^+)^{1/20}} \geq e^{\frac{8\sqrt{\log b}}{3}(y_j)^{1/20}}.$$

Fix  $j \geq 1$  and  $y_j$  with  $y_j^- \leq y_j \leq y_j^+$ . Recall (5.29) and the interval (5.30). Fix  $v^* \in T_{n-\ell_j}$ . Let  $\tilde{L}^\downarrow$  be a local time of a Brownian motion on  $\tilde{T}_{\leq \ell_j}^{v^*}$  and set  $\tilde{\tau}(s) := \inf\{r \geq 0 : \tilde{L}_r^\downarrow(v^*) > s\}$ . Recall  $\tilde{P}_{n-\ell_j}^B$  and  $\tilde{B}$  defined in (3.14) and (3.15). We have for  $t \geq n \log n$

$$\begin{aligned}
& b^{-n+\ell_j} \tilde{E}_\rho \left[ \Lambda_{y_j, \ell_j, J_{\ell_j}}^n(t) \right] \\
&= \tilde{E}_\rho \left[ 1_{\left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v^*)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + y_j, \text{ for all } 0 \leq s \leq n - \ell_j, \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v^*)} \in \sqrt{t} + \frac{a_n(t)}{n} (n - \ell_j) + J_{\ell_j} \right\}} \right. \\
&\quad \left. \cdot \tilde{P}_{v^*} \left( \max_{u \in T_{\ell_j}^{v^*}} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^\downarrow(\tilde{L}_{\tilde{\tau}(t)}^n(v^*))}(u)} > \sqrt{t} + a_n(t) + y_j \right) \right] \\
&= \left( 1 + O\left(\frac{1}{\log n}\right) \right) E_0^B \left[ 1_{\left\{ -(1-\delta)\sqrt{t} \leq B_s/\sqrt{2} \leq \frac{a_n(t)}{n} s + y_j, \text{ for all } 0 \leq s \leq n - \ell_j \right\}} \right. \\
&\quad \left. 1_{\left\{ B_{n-\ell_j}/\sqrt{2} \in \frac{a_n(t)}{n} (n - \ell_j) + J_{\ell_j} \right\}} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + B_{n-\ell_j}/\sqrt{2}}} \right. \\
&\quad \left. \cdot \tilde{P}_{v^*} \left( \max_{u \in T_{\ell_j}^{v^*}} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^\downarrow \left( \left( \sqrt{t} + B_{n-\ell_j}/\sqrt{2} \right)^2 \right)}(u)} > \sqrt{t} + a_n(t) + y_j \right) \right] \\
&= \left( 1 + O\left(\frac{1}{\log n}\right) \right) \tilde{E}_{n-\ell_j}^B \left[ 1_{\left\{ \tilde{B}_s/\sqrt{2} \leq y_j, \text{ for all } 0 \leq s \leq n - \ell_j, \tilde{B}_{n-\ell_j}/\sqrt{2} \in J_{\ell_j} \right\}} \phi\left(\frac{\tilde{B}_{n-\ell_j}}{\sqrt{2}}\right) \right] \\
&\quad - \left( 1 + O\left(\frac{1}{\log n}\right) \right) \tilde{E}_{n-\ell_j}^B \left[ 1_{\left\{ \tilde{B}_s/\sqrt{2} \leq y_j, \text{ for all } 0 \leq s \leq n - \ell_j, \tilde{B}_{n-\ell_j}/\sqrt{2} \in J_{\ell_j} \right\}} \right. \\
&\quad \left. \cdot 1_{\left\{ \tilde{B}_s/\sqrt{2} < -(1-\delta)\sqrt{t} - \frac{a_n(t)}{n} s, \text{ for some } 0 \leq s \leq n - \ell_j \right\}} \phi\left(\frac{\tilde{B}_{n-\ell_j}}{\sqrt{2}}\right) \right] \\
&=: K_1 - K_2, \tag{5.35}
\end{aligned}$$

where

$$\begin{aligned}
\phi(x) &:= e^{-\frac{2a_n(t)}{n}x - \frac{(a_n(t))^2}{n^2}(n-\ell_j)} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell_j) + x}} \\
&\quad \cdot \tilde{P}_{v^*} \left( \max_{u \in T_{\ell_j}^{v^*}} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^\downarrow \left( \left( \sqrt{t} + \frac{a_n(t)}{n} (n - \ell_j) + x \right)^2 \right)}(u)} > \sqrt{t} + a_n(t) + y_j \right),
\end{aligned}$$

and we have used Lemma 2.1 in the first equality, and in the second equality, Lemma 2.5, (2.7), and the fact that for  $t \geq n \log n$ ,  $\exp\left(-\frac{3}{8} \int_0^{n-\ell_j} \frac{ds}{X_s}\right) = 1 + O((\log n)^{-1})$  under the event that  $\sqrt{X_s/2} \geq \delta \sqrt{t}$  for all  $0 \leq s \leq n - \ell_j$ .

We first estimate  $K_2$ . For all  $\varepsilon > 0$  and  $j \geq 1$ , there exists  $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$  such that for all  $n \geq n_0(j)$ , we have

$$K_2 \leq c_1 b^{-(n-\ell_j)} \sqrt{\ell_j} n^{1-(1-\delta)^2 c_*} y_j e^{-2\sqrt{\log b} y_j} \leq \varepsilon b^{-(n-\ell_j)} y_j e^{-2\sqrt{\log b} y_j}, \tag{5.36}$$

uniformly in  $y_j$  and  $t$  satisfying (5.2) (we take  $c_*$  large enough), where we have used (3.25) and Proposition 3.1(i). Next, we estimate  $K_1$ . By the density (2.12), we have

$$\begin{aligned}
K_1 &= \left(1 + O\left(\frac{1}{\log n}\right)\right) \\
&\quad \cdot \int_{(\ell_j)^{2/5}}^{\ell_j} \frac{1}{\sqrt{\pi(n-\ell_j)}} \left( e^{-\frac{(z-y_j)^2}{n-\ell_j}} - e^{-\frac{(z+y_j)^2}{n-\ell_j}} \right) \\
&\quad \quad e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}y_j - \frac{(a_n(t))^2}{n^2}(n-\ell_j)} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell_j) + y_j - z}} \\
&\quad \quad \cdot \tilde{P}_{v^*} \left( \max_{u \in T_{\ell_j}^{v^*}} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow \left( \left( \sqrt{t} + \frac{a_n(t)}{n}(n-\ell_j) + y_j - z \right)^2 \right)}(u) > \sqrt{t} + a_n(t) + y_j \right) dz \\
&= b^{-(n-\ell_j)} \left(1 + O\left(\frac{1}{\log n}\right)\right) \frac{4}{\sqrt{\pi}} y_j e^{-2\sqrt{\log b} y_j} \\
&\quad \cdot \int_{(\ell_j)^{2/5}}^{\ell_j} z e^{2\sqrt{\log b} z} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell_j) + y_j - z}} \\
&\quad \cdot P_\rho \left( \frac{\max_{u \in T_{\ell_j}} L_{\tau(s_{z,y_j,\ell_j}^n(t))}^{\ell_j}(u) - s_{z,y_j,\ell_j}^n(t)}{\sqrt{2s_{z,y_j,\ell_j}^n(t)}} > \sqrt{2} \left( \sqrt{\log b} \ell_j + z \right) + \Delta_{z,y_j,\ell_j}^n(t) \right) dz,
\end{aligned} \tag{5.37}$$

where we have set

$$s_{z,y_j,\ell_j}^n(t) := \left( \sqrt{t} + \frac{a_n(t)}{n}(n-\ell_j) + y_j - z \right)^2$$

and  $\Delta_{z,y_j,\ell_j}^n(t)$  as the remainder term, and used the fact that the law of the Brownian motion on  $\tilde{T}_{\leq \ell_j}^{v^*}$  starting at  $v^*$  is the same as that of the Brownian motion on  $\tilde{T}_{\leq \ell_j}$  starting at  $\rho$ . One can show that for each  $j \geq 1$ ,

$$\lim_{n \rightarrow \infty} \Delta_{z,y_j,\ell_j}^n(t) = 0, \quad \text{uniformly in } z \in [(\ell_j)^{2/5}, \ell_j] \text{ and } y_j, t \text{ satisfying (5.2)}. \tag{5.38}$$

By Theorem 2.4(i), (2.3), and (2.4), we have for any fixed  $m \in \mathbb{N}$

$$\left( \frac{L_{\tau(s)}^m(v) - s}{\sqrt{2s}} \right)_{v \in T_m} \longrightarrow (h_v)_{v \in T_m} \quad \text{in law as } s \rightarrow \infty.$$

By this, the definition of (5.34), and (5.38), for all  $\varepsilon > 0$ , there exists  $j_0 \in \mathbb{N}$  such that the following holds: for each  $j \geq j_0$ , there exists  $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$  such that for all  $n \geq n_0(j)$ ,

$$\left| \int_{(\ell_j)^{2/5}}^{\ell_j} P_\rho \left( \frac{\max_{u \in T_{\ell_j}} L_{\tau(s_{z,y_j,\ell_j}^n(t))}^{\ell_j}(u) - s_{z,y_j,\ell_j}^n(t)}{\sqrt{2s_{z,y_j,\ell_j}^n(t)}} > \sqrt{2} \left( \sqrt{\log b} \ell_j + z \right) + \Delta_{z,y_j,\ell_j}^n(t) \right) dz \right|$$



$$\left| \int z e^{2\sqrt{\log b} z} dz - \tilde{\gamma}_* \right| < \varepsilon, \quad (5.39)$$

uniformly in  $y_j$  and  $t$  satisfying (5.2). Thus, by (5.35)-(5.37), (5.39), and the definition (1.8) of  $\beta_*$ , for all  $\varepsilon > 0$ , there exists  $j_0 \in \mathbb{N}$  such that the following holds: for each  $j \geq j_0$ , there exists  $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$  such that for all  $n \geq n_0(j)$ ,

$$\left| \frac{\tilde{E}_\rho \left[ \Lambda_{y_j, \ell_j, \ell_j}^n(t) \right]}{y_j e^{-2\sqrt{\log b} y_j}} - \frac{4}{\sqrt{\pi}} \beta_* \tilde{\gamma}_* \right| < \varepsilon, \quad (5.40)$$

uniformly in  $y_j$  and  $t$  satisfying (5.2). By Lemma 5.4, 5.5, and (5.40), for all  $\varepsilon > 0$ , there exists  $j_0 \in \mathbb{N}$  such that the following holds: for each  $j \geq j_0$ , there exists  $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$  such that for all  $n \geq n_0(j)$ , (5.1) holds by replacing  $\gamma_*$  with  $\tilde{\gamma}_*$  uniformly in  $y_j$  and  $t$  satisfying (5.2) (we take  $c_* > 0$  large enough). Let  $\hat{\gamma}_*$  be the limit of any convergent subsequence of (5.33). By taking a sub-subsequence, if necessary, and repeating the above argument, we have (5.1) if we replace  $\gamma_*$  with  $\hat{\gamma}_*$ . Thus, the full sequence (5.33) converges to a finite positive constant, and we write  $\gamma_*$  to denote the limit. Therefore, we have (5.1).  $\square$

## 6 Proof of Theorem 1.1 and Corollary 1.3

In this section, we prove Theorem 1.1 and Corollary 1.3. We begin with preliminary lemmas. Let  $(h_v)_{v \in T}$  be a BRW on  $T$  defined in Section 1. For each  $n \in \mathbb{N}$ , we set

$$D_n^{(2)} := \sum_{v \in T_n} \left( \sqrt{\log b} n - \frac{1}{\sqrt{2}} h_v \right)^2 e^{-4\sqrt{\log b} \left( \sqrt{\log b} n - \frac{1}{\sqrt{2}} h_v \right)}.$$

Then the following holds:

**Lemma 6.1** *For all  $\varepsilon > 0$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( D_n^{(2)} \geq \varepsilon \right) = 0. \quad (6.1)$$

*Proof.* Set  $m_n := \sqrt{\log b} n - \frac{3}{4\sqrt{\log b}} \log n$ . We have for any  $y > 0$

$$\begin{aligned} & \mathbb{P} \left( D_n^{(2)} \geq \varepsilon \right) \\ & \leq \mathbb{P} \left( \sum_{v \in T_n} \left( \sqrt{\log b} n - h_v / \sqrt{2} \right)^2 e^{-4\sqrt{\log b} \left( \sqrt{\log b} n - h_v / \sqrt{2} \right)} 1_{\{h_v / \sqrt{2} \leq m_n + y\}} \geq \varepsilon \right) \\ & \quad + \mathbb{P} \left( \max_{v \in T_n} h_v / \sqrt{2} > m_n + y \right). \end{aligned} \quad (6.2)$$

Since  $h_v$  is a Gaussian random variable with mean 0 and variance  $n$  for each  $v \in T_n$ , a simple calculation implies that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \sum_{v \in T_n} \left( \sqrt{\log b} \, n - h_v / \sqrt{2} \right)^2 e^{-4\sqrt{\log b} \left( \sqrt{\log b} \, n - h_v / \sqrt{2} \right)} 1_{\{h_v / \sqrt{2} \leq m_n + y\}} \right] = 0.$$

By this and applying the Chebyshev inequality to the first term of (6.2) and Lemma A.5(i) to the second term of (6.2), we have

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( D_n^{(2)} \geq \varepsilon \right) \leq c_1 (1 + y) e^{-2\sqrt{\log b} \, y}.$$

Since we can take arbitrary  $y \in (0, \infty)$ , this implies (6.1).  $\square$

For  $a \in \mathbb{R}$ , an interval  $I \subset \mathbb{R}$ ,  $t > 0$ , and  $r, n \in \mathbb{N}$  with  $r < n$ , we set

$$\mathcal{B}_{r,t,n}^\sigma(I; a) := \left\{ \sqrt{L_{\tau(t)}^n(v)} - \sqrt{t} - a_n(t) \notin I, \right. \\ \left. \text{for all } v \in T_n \text{ with } \sigma(v_r) \in [a - b^{-r}, a] \right\}. \quad (6.3)$$

Then we have the following:

**Lemma 6.2** *There exists  $c_1 > 0$  such that the following holds: for any finite interval  $I \subset \mathbb{R}$ , there exist  $c_2(I) > 0$  and  $r_0(I) \in \mathbb{N}$  such that for all  $t > 0$ ,  $a \in \mathbb{R}$ ,  $r \geq r_0(I)$ , and  $n > r$ ,*

$$P_\rho \left[ \left( \mathcal{B}_{r,t,n}^\sigma(I; a) \right)^c \right] \leq c_2(I) r^3 e^{-c_1 r}. \quad (6.4)$$

*Proof.* Fix any finite interval  $I \subset \mathbb{R}$ . Let  $\{\bar{I}, \underline{I}\}$  be the boundary of  $I$  with  $\underline{I} < \bar{I}$ . Fix any  $t > 0$ ,  $a \in \mathbb{R}$ ,  $r \geq r_0(I)$ , and  $n > r$ , where we take  $r_0(I) > |\bar{I}|$  large enough. Recall the event  $G_r^n(t)$  defined in (3.3). We have

$$\begin{aligned} & P_\rho \left[ \left( \mathcal{B}_{r,t,n}^\sigma(I; a) \right)^c \right] \\ & \leq \tilde{P}_\rho \left( \text{there exists } v \in T_n \text{ with } \sigma(v_r) \in [a - b^{-r}, a] \text{ such that} \right. \\ & \quad \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + \kappa(\log(s \wedge (n-s)))_+ + r + 1, \\ & \quad \left. \text{for all } 0 \leq s \leq n, \sqrt{\tilde{L}_{\tau(t)}^n(v)} - \sqrt{t} - a_n(t) \in I \right) \\ & \quad + \tilde{P}_\rho(G_r^n(t)). \end{aligned} \quad (6.5)$$

By a simple observation, we have

$$\left| \{v \in T_n : \sigma(v_r) \in [a - b^{-r}, a]\} \right| \leq c_1 r b^{n-r}. \quad (6.6)$$

By (6.6) and an argument similar to (3.21), the first term on the right-hand side of (6.5) is bounded from above by  $c_2(I) b^{-r} r^3$ . By this and Lemma 3.2, we have (6.4).  $\square$

*Proof of Theorem 1.1.* Recall (1.4), (1.5), (1.8), and (1.9). For each interval  $I \subset \mathbb{R}$ , we will write  $\partial I$  to denote its boundary. Fix any sequence of positive integers  $(r_n)_{n \geq 1}$

with  $\lim_{n \rightarrow \infty} r_n = \infty$  and  $\lim_{n \rightarrow \infty} r_n/n \in [0, 1]$ . Fix any  $(t_n)_{n \geq 1}$  with  $\lim_{n \rightarrow \infty} \sqrt{t_n}/n = \theta \in [0, \infty]$  and  $t_n \geq c_* n \log n$ , where  $c_*$  is a sufficiently large positive constant. We will prove the following: for all finite disjoint intervals  $A_i := (\underline{A}_i, \overline{A}_i] \subset [0, 1]$ ,  $1 \leq i \leq m$  with  $\mathbb{P}(Z_\infty(\partial A_i) = 0) = 1$  for all  $1 \leq i \leq m$ , finite intervals  $B_i := (\underline{B}_i, \overline{B}_i]$ ,  $1 \leq i \leq m$ , and positive values  $a_1, \dots, a_m$ ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} E_\rho \left[ \exp \left\{ - \sum_{i=1}^m a_i \Xi_{n, t_n}^{(n-r_n)} (A_i \times B_i) \right\} \right] \\ &= \mathbb{E} \left[ \exp \left\{ - \frac{4}{\sqrt{\pi}} \beta_* \gamma_* \sum_{i=1}^m (1 - e^{-a_i}) Z_\infty(A_i) \left( e^{-2\sqrt{\log b} \underline{B}_i} - e^{-2\sqrt{\log b} \overline{B}_i} \right) \right\} \right]. \end{aligned} \quad (6.7)$$

Let  $q < n$  be a positive integer. Recall the events (6.3). We have

$$\begin{aligned} & E_\rho \left[ \exp \left\{ - \sum_{i=1}^m a_i \Xi_{n, t_n}^{(q)} (A_i \times B_i) \right\} \right] \\ &= E_\rho \left[ \exp \left\{ - \sum_{i=1}^m a_i \Xi_{n, t_n}^{(q)} (A_i \times B_i) \right\} 1_{\cap_{i=1}^m \mathcal{B}_{q, t_n, n}^\sigma(B_i; \underline{A}_i) \cap \mathcal{B}_{q, t_n, n}^\sigma(B_i; \overline{A}_i)} \right] \\ &+ E_\rho \left[ \exp \left\{ - \sum_{i=1}^m a_i \Xi_{n, t_n}^{(q)} (A_i \times B_i) \right\} 1_{\cup_{i=1}^m (\mathcal{B}_{q, t_n, n}^\sigma(B_i; \underline{A}_i))^c \cup (\mathcal{B}_{q, t_n, n}^\sigma(B_i; \overline{A}_i))^c} \right] \\ &=: J_1 + J_2, \end{aligned} \quad (6.8)$$

We estimate  $J_1$  in (6.8). Under the event  $\cap_{i=1}^m \mathcal{B}_{q, t_n, n}^\sigma(B_i; \overline{A}_i) \cap \mathcal{B}_{q, t_n, n}^\sigma(B_i; \underline{A}_i)$ , we have for all  $v \in T_q$  and  $1 \leq i \leq m$ ,

$$\begin{aligned} & 1_{\left\{ \sigma(\arg \max_v L_{\tau(t_n)}^n) \in A_i, \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} - \sqrt{t_n} - a_n(t_n) \in B_i \right\}} \\ &= 1_{\left\{ \sigma(v) \in A_i, \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} - \sqrt{t_n} - a_n(t_n) \in B_i \right\}}. \end{aligned} \quad (6.9)$$

Thus, we have

$$\begin{aligned} J_1 &= E_\rho \left[ \exp \left\{ - \sum_{v \in T_q} \sum_{i=1}^m a_i 1_{\left\{ \sigma(v) \in A_i, \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} - \sqrt{t_n} - a_n(t_n) \in B_i \right\}} \right\} \right] \\ &- E_\rho \left[ \exp \left\{ - \sum_{v \in T_q} \sum_{i=1}^m a_i 1_{\left\{ \sigma(v) \in A_i, \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} - \sqrt{t_n} - a_n(t_n) \in B_i \right\}} \right\} \right. \\ &\quad \left. 1_{\cup_{i=1}^m (\mathcal{B}_{q, t_n, n}^\sigma(B_i; \underline{A}_i))^c \cup (\mathcal{B}_{q, t_n, n}^\sigma(B_i; \overline{A}_i))^c} \right] \\ &=: J_{1,1} - J_{1,2}. \end{aligned} \quad (6.10)$$

By Lemma 6.2, we have

$$\max\{J_2, J_{1,2}\} \leq \sum_{i=1}^m c_1(B_i) q^3 e^{-c_2 q}. \quad (6.11)$$

We estimate  $J_{1,1}$ . By Theorem 2.4(ii), on the same probability space (we will write  $P$  to denote the probability measure), we can construct a local time  $(L_{\tau(t_n)}^n(v))_{v \in T_{\leq n}}$  and two BRWs  $(h_v)_{v \in T_{\leq n}}, (h'_v)_{v \in T_{\leq n}}$  satisfying (2.5) and (2.6). Fix  $\delta \in (0, 1)$ . We set

$$C_q := \left\{ \frac{1}{\sqrt{2}}|h_v|, \frac{1}{\sqrt{2}}|h'_v| \in \left[ 0, \sqrt{\log b} q - \frac{3}{4\sqrt{\log b}}(1-\delta)\log q \right], \text{ for all } v \in T_q \right\}. \quad (6.12)$$

For each  $v \in T_q$ , let  $\tilde{L}^\downarrow$  be a local time of a Brownian motion on  $\tilde{T}_{\leq n-q}^v$  and set  $\tilde{\tau}^\downarrow(p) := \inf\{s \geq 0 : \tilde{L}_s^\downarrow(v) > p\}$ . We omit the subscript  $v$  in  $\tilde{L}^\downarrow$  and  $\tilde{\tau}^\downarrow$ . By Lemma 2.1, we have

$$J_{1,1} = E \left[ 1_{C_q} \prod_{v \in T_q} K_v \right] + E \left[ 1_{(C_q)^c} \prod_{v \in T_q} K_v \right] =: J_{1,1,1} + J_{1,1,2}, \quad (6.13)$$

where for each  $v \in T_q$ , we have set

$$K_v := \tilde{E}_v \left[ \exp \left\{ - \sum_{i=1}^m a_i 1 \left\{ \sigma(v) \in A_i, \max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t_n)}^n(v))}^{(u)} - \sqrt{t_n} - a_n(t_n) \in B_i \right\} \right\} \right]. \quad (6.14)$$

By Lemma A.5, we have

$$J_{1,1,2} \leq c_3(\log q) \cdot (q)^{-\frac{3}{2}\delta}. \quad (6.15)$$

We estimate  $J_{1,1,1}$ . Since  $A_1, \dots, A_m$  are disjoint, we have for all  $v \in T_q$ ,

$$K_v = \exp \left\{ \log \left( 1 - \sum_{i=1}^m (1 - e^{-a_i}) 1_{\{\sigma(v) \in A_i\}} \cdot \tilde{P}_v \left( \max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t_n)}^n(v))}^{(u)} - \sqrt{t_n} - a_n(t_n) \in B_i \right) \right) \right\}. \quad (6.16)$$

On the event  $C_q$ , by (2.6), we have for all  $v \in T_q$

$$\sqrt{t_n} + a_n(t_n) = \sqrt{L_{\tau(t_n)}^n(v) + a_{n-q}} \left( L_{\tau(t_n)}^n(v) \right) + \sqrt{\log b} q - \frac{1}{\sqrt{2}} h'_v + \delta_{v,q}^n, \quad (6.17)$$

where

$$\delta_{v,q}^n := \frac{3 \log(1 - q/n)}{4\sqrt{\log b}} + \frac{\log(1 + O(q/\sqrt{t_n}))}{4\sqrt{\log b}} + O(q^2/\sqrt{t_n}). \quad (6.18)$$

We take any sufficiently small  $\varepsilon > 0$  and sufficiently large  $q_0 \in \mathbb{N}$  which depends on  $B_1, \dots, B_m$  and  $\varepsilon$ . We assume that  $q \geq q_0$  and  $n \geq n_0$ , where we take sufficiently large  $n_0 = n_0(q, \varepsilon) \in \mathbb{N}$ . By Proposition 5.1 and (6.17)-(6.18), under the event  $C_q$ , we have for all  $1 \leq i \leq m$  and  $v \in T_q$ ,

$$\tilde{P}_v \left( \max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t_n)}^n(v))}^{(u)} - \sqrt{t_n} - a_n(t_n) \in B_i \right)$$

$$\begin{aligned}
&= \tilde{P}_v \left( \max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t_n)}^n(v))}(u) - \sqrt{L_{\tau(t_n)}^n(v)} - a_{n-q} \left( L_{\tau(t_n)}^n(v) \right) \right. \\
&\quad \left. \geq \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \underline{B}_i + \delta_{v,q}^n \right) \\
&- \tilde{P}_v \left( \max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t_n)}^n(v))}(u) - \sqrt{L_{\tau(t_n)}^n(v)} - a_{n-q} \left( L_{\tau(t_n)}^n(v) \right) \right. \\
&\quad \left. \geq \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \bar{B}_i + \delta_{v,q}^n \right) \\
&\geq (1 - c_4 \varepsilon) \frac{4}{\sqrt{\pi}} \beta_* \gamma_* \left( \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \underline{B}_i \right) e^{-2\sqrt{\log b} \left( \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \underline{B}_i \right)} \\
&\quad - (1 + c_4 \varepsilon) \frac{4}{\sqrt{\pi}} \beta_* \gamma_* \left( \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \bar{B}_i \right) e^{-2\sqrt{\log b} \left( \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \bar{B}_i \right)}. \quad (6.19)
\end{aligned}$$

We set

$$\begin{aligned}
D_q^{(2)} &:= \sum_{v \in T_q} \left( \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v \right)^2 e^{-4\sqrt{\log b} \left( \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v \right)}, \\
W_q &:= \sum_{v \in T_q} e^{-2\sqrt{\log b} \left( \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v \right)}.
\end{aligned}$$

Recall the random measure  $Z_q$  defined in (1.3). By (6.16), (6.19) and Taylor's expansion of the function  $x \mapsto \log(1-x)$ , under the event  $C_q \cap \{D_q^{(2)} < \varepsilon\} \cap \{W_q < \varepsilon\}$ , we have

$$\begin{aligned}
\prod_{v \in T_q} K_v &\leq e^{c_5(B_1, \dots, B_m)\varepsilon} \cdot \exp \left\{ -\frac{4}{\sqrt{\pi}} \beta_* \gamma_* \sum_{i=1}^m (1 - e^{-a_i}) Z_q(A_i) \right. \\
&\quad \left. \cdot \left( e^{-2\sqrt{\log b} \, \underline{B}_i} - e^{-2\sqrt{\log b} \, \bar{B}_i} - c_4 \varepsilon \left( e^{-2\sqrt{\log b} \, \underline{B}_i} + e^{-2\sqrt{\log b} \, \bar{B}_i} \right) \right) \right\}, \quad (6.20)
\end{aligned}$$

We can obtain a similar lower bound of  $\prod_{v \in T_q} K_v$ . By Lemma A.5, Lemma 6.1, and the fact that  $\lim_{q \rightarrow \infty} W_q = 0$ , almost surely (see [32]), we have

$$\lim_{q \rightarrow \infty} P \left( (C_q)^c \cup \{D_q^{(2)} \geq \varepsilon\} \cup \{W_q \geq \varepsilon\} \right) = 0. \quad (6.21)$$

Thus, by (6.8), (6.10)-(6.11), (6.13), (6.15) (6.20) (and a similar lower bound), and (6.21), taking  $n \rightarrow \infty$ , then  $q \rightarrow \infty$ , and finally  $\varepsilon \rightarrow 0$ , we have

$$\begin{aligned}
&\lim_{q \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| E_\rho \left[ \exp \left\{ -\sum_{i=1}^m a_i \Xi_{n,t_n}^{(q)}(A_i \times B_i) \right\} \right] \right. \\
&\quad \left. - \mathbb{E} \left[ \exp \left\{ -\frac{4}{\sqrt{\pi}} \beta_* \gamma_* \sum_{i=1}^m (1 - e^{-a_i}) Z_\infty(A_i) \left( e^{-2\sqrt{\log b} \, \underline{B}_i} - e^{-2\sqrt{\log b} \, \bar{B}_i} \right) \right\} \right] \right| = 0. \quad (6.22)
\end{aligned}$$

Next, by using (6.22), we will prove (6.7). Let  $z_*$  be a real number with  $z_* < \min_{1 \leq i \leq m} \underline{B}_i$ . Take  $q_0 = q_0(z_*) \in \mathbb{N}$  large enough and fix any  $q \geq q_0$ . Take  $n \in \mathbb{N}$  large enough so that  $q < n - r_n < n - q$  and  $q < n/4$ . We set

$$U_{z_*, q}^n(t_n) := \left\{ \text{there exist } v, u \in T_n \text{ with } q \leq |v \wedge u| \leq n - q \text{ such that} \right. \\ \left. \sqrt{L_{\tau(t_n)}^n(v)}, \sqrt{L_{\tau(t_n)}^n(u)} \in [\sqrt{t_n} + a_n(t_n) + z_*, \infty) \right\}.$$

Under the event  $(U_{z_*, q}^n(t_n))^c$ , by the definition of  $\arg \max_v L_{\tau(t_n)}^n$ , we have

$$\left\{ \arg \max_v L_{\tau(t_n)}^n : v \in T_{n-r_n}, \max_{u \in T_{n-r_n}^v} \sqrt{L_{\tau(t_n)}^n(u)} \geq \sqrt{t_n} + a_n(t_n) + z_* \right\} \\ = \left\{ \arg \max_v L_{\tau(t_n)}^n : v \in T_q, \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} \geq \sqrt{t_n} + a_n(t_n) + z_* \right\}. \quad (6.23)$$

By (6.22), (6.23),  $z_* < \min_{1 \leq i \leq m} \underline{B}_i$ , and Proposition 4.1, we have (6.7) which implies that the point process  $\Xi_{n, t_n}^{(n-r_n)}$  converges in law to the Cox process (1.7) as  $n \rightarrow \infty$  (see, for example, [20, Proposition 11.1.VIII]).  $\square$

*Proof of Corollary 1.3.* Corollary 1.3 immediately follows from Theorem 1.1 and Proposition 3.1(i). We omit the details.  $\square$

## A Appendix

In this section, we give proof of some technical estimates.

### A.1 Proof of Lemma 2.1

*Proof of Lemma 2.1.* Following [26, Section 4], we will construct a version of the Brownian motion on  $\tilde{T}_{\leq n}$  via piecing together excursions around  $a$ . Let  $\tilde{X}^0 = (\tilde{X}_t^0, t \geq 0, \tilde{P}_x^0, x \in \tilde{T}_{\leq n} \setminus \{a\})$  be a Brownian motion on  $\tilde{T}_{\leq n} \setminus \{a\}$  killed upon the hitting time of  $a$ . We define a space of excursions around  $a$  by

$$W_a := \{w : \text{there exists } \xi(w) \in (0, \infty) \text{ such that } w : [0, \xi(w)] \rightarrow \tilde{T}_{\leq n} \text{ is continuous,} \\ w(t) \in \tilde{T}_{\leq n} \setminus \{a\} \text{ for all } t \in (0, \xi(w)), \text{ and } w(0) = w(\xi(w)) = a\}.$$

Let  $\sum_{s \in D} \delta_{(s, \mathbf{e}_s)}$  be a Poisson point process on  $(0, \infty) \times W_a$  with intensity measure  $ds \times \mathbf{n}$  defined on a probability space  $(\Omega, P)$ , where the set  $D \subset (0, \infty)$  is countable and  $\mathbf{n}$  is the excursion law corresponding to  $\tilde{X}^0$ . See [26, Section 4] for the detail. Let  $\partial$  be a new point and we extend  $\mathbf{n}$  to a measure on  $W_a \cup \{\partial\}$  by setting  $\mathbf{n}(\{\partial\}) = 0$ . For  $s \notin D$ , we set  $\mathbf{e}_s := \partial$  and  $\xi(\mathbf{e}_s) = 0$ . Set  $J(s) := \sum_{r \leq s} \xi(\mathbf{e}_r)$ . For  $t \geq 0$ , we can find  $s$  with

$J(s-) \leq t \leq J(s)$  and set

$$Y_t := \begin{cases} \mathbf{e}_s(t - J(s-)) & \text{if } J(s-) < J(s), \\ a & \text{if } J(s-) = J(s). \end{cases} \quad (\text{A.1})$$

Let  $\Omega^0$  be the sample space on which  $\tilde{X}^0$  is defined. We assume that  $\Omega^0$  has an extra point  $\omega^a$  with  $\tilde{P}_x(\{\omega^a\}) = 0$  for all  $x \in \tilde{T}_{\leq n} \setminus \{a\}$ . Set  $\tilde{P}_a^0 := \delta_{\omega^a}$ . We define the product probability space  $(\tilde{\Omega}, \tilde{P}_x)$  by

$$\tilde{\Omega} := \Omega^0 \times \Omega, \quad \tilde{P}_x := \tilde{P}_x^0 \times P, \quad x \in \tilde{T}_{\leq n}.$$

We define a new process  $\tilde{X} = (\tilde{X}_t, t \geq 0, \tilde{P}_x, x \in \tilde{T}_{\leq n})$  as follows: for  $\tilde{\omega} = (\omega^0, \omega) \in \Omega^0 \setminus \{\omega^a\} \times \Omega$ , we set

$$\tilde{X}_t(\tilde{\omega}) := \begin{cases} \tilde{X}_t^0(\omega^0) & \text{if } 0 \leq t < H_a^0(\omega^0), \\ Y_{t-H_a^0(\omega^0)}(\omega) & \text{if } t \geq H_a^0(\omega^0), \end{cases} \quad (\text{A.2})$$

where  $H_a^0 := \inf\{t \geq 0 : \tilde{X}_t^0 = a\}$ . For  $\tilde{\omega} = (\omega^a, \omega)$  with  $\omega \in \Omega$ , we set  $\tilde{X}_t(\tilde{\omega}) := Y_t(\omega)$  for all  $t \geq 0$ . By [26, Theorem 4.1 and Corollary 5.1],  $\tilde{X}$  is a Brownian motion on  $\tilde{T}_{\leq n}$ . We consider the space of excursions  $W_a^\uparrow$  restricted on  $\tilde{T}_{\leq n} \setminus \tilde{T}_{\leq n-|a|}^a$  defined by

$$W_a^\uparrow := \{w \in W_a : w(t) \in \tilde{T}_{\leq n} \setminus \tilde{T}_{\leq n-|a|}^a \text{ for all } t \in (0, \xi(w))\}.$$

Similarly, we define the space of excursions  $W_a^\downarrow$  restricted on  $\tilde{T}_{\leq n-|a|}^a$ . For  $s \geq 0$ , set

$$\mathbf{e}_s^\uparrow := \begin{cases} \mathbf{e}_s & \text{if } \mathbf{e}_s \in W_a^\uparrow, \\ \partial & \text{otherwise,} \end{cases} \quad \mathbf{e}_s^\downarrow := \begin{cases} \mathbf{e}_s & \text{if } \mathbf{e}_s \in W_a^\downarrow, \\ \partial & \text{otherwise.} \end{cases}$$

Since  $W_a^\uparrow$  and  $W_a^\downarrow$  are disjoint, two Poisson point processes  $\sum_s \delta_{(s, \mathbf{e}_s^\uparrow)}$  and  $\sum_s \delta_{(s, \mathbf{e}_s^\downarrow)}$  are independent. For  $s \geq 0$ , set  $J^\uparrow(s) := \sum_{r \leq s} \xi(\mathbf{e}_r^\uparrow)$  and  $J^\downarrow(s) := \sum_{r \leq s} \xi(\mathbf{e}_r^\downarrow)$ . Let  $\{\tilde{L}_t^n(x) : (t, x) \in [0, \infty) \times \tilde{T}_{\leq n}\}$  be a local time of  $\tilde{X}$ . For  $t > 0$  and  $\tilde{\omega} = (\omega^0, \omega) \in \tilde{\Omega}$ , we set

$$\hat{L}_t^n(a)(\tilde{\omega}) := \inf\{s : H_a^0(\omega^0) + J(s)(\omega) > t\}.$$

One can check that  $\{\hat{L}_t^n(a) : t \geq 0\}$  is a local time of  $\tilde{X}$  at  $a$ . Thus, by [35, Lemma 3.6.8],  $\hat{L}_t^n(a) = \tilde{L}_t^n(a)$  for all  $t \geq 0$ ,  $\tilde{P}_\rho$ -almost surely. By [27, Theorem 2.6] and [34, Theorem 3.7],  $\tilde{X}$  has a continuous 1-potential density. Thus, by [35, Theorem 3.6.3], for each  $y \in \tilde{T}_{\leq n}$ , there exists a positive sequence  $(\varepsilon_m^y)_{m \geq 0}$  with  $\lim_{m \rightarrow \infty} \varepsilon_m^y = 0$  such that  $\tilde{P}_\rho$ -almost surely,

$$\tilde{L}_t^n(y) = \lim_{m \rightarrow \infty} \int_0^t f_{\varepsilon_m^y}(\tilde{X}_s) ds, \quad \text{locally uniformly in } t, \quad (\text{A.3})$$

where  $f_{\varepsilon,y}$  is an approximate  $\delta$ -function at  $y$  with respect to  $m$ : that is,  $\{f_{\varepsilon,y} : \varepsilon > 0\}$  is a family of positive continuous functions on  $\tilde{T}_{\leq n}$  satisfying  $\int_{\tilde{T}_{\leq n}} f_{\varepsilon,y}(x) m(dx) = 1$  (recall the measure  $m$  defined in Section 2) and  $\text{supp}(f_{\varepsilon,y})$  is a compact neighborhood of  $y$  with  $\text{supp}(f_{\varepsilon,y}) \downarrow \{y\}$  as  $\varepsilon \rightarrow 0$ . By (A.3), we have

$$\begin{aligned} \tilde{L}_{\tilde{\tau}(t)}^n(a) &= \inf\{s : \tilde{L}_{H_a^0 + J(s)}^n(\rho) > t\} \\ &= \inf\left\{s : \lim_{m \rightarrow \infty} \left( \int_0^{H_a^0} f_{\varepsilon_m, \rho}(\tilde{X}_u^0) du + \sum_{r \leq s} \int_0^{\xi(\mathbf{e}_r^\uparrow)} f_{\varepsilon_m, \rho}(\mathbf{e}_r^\uparrow(u)) du \right) > t \right\} \\ &=: s_*^n(t), \text{ a.s.}, \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} \tilde{\tau}(t) &= \inf\{s : \tilde{L}_s^n(\rho) > t\} \\ &= \inf\left\{s : \lim_{m \rightarrow \infty} \left( \int_0^{H_a^0} f_{\varepsilon_m, \rho}(X_u^0) du + \sum_{r < s_*^n(t)} \int_0^{\xi(\mathbf{e}_r^\uparrow)} f_{\varepsilon_m, \rho}(\mathbf{e}_r^\uparrow(u)) du \right. \right. \\ &\quad \left. \left. + \int_0^{s - J^\uparrow(s_*^n(t) -)} f_{\varepsilon_m, \rho}(\mathbf{e}_{s_*^n(t)}^\uparrow(u)) du \right) > t \right\}, \text{ a.s.}, \end{aligned} \quad (\text{A.5})$$

$$\tilde{L}_{\tilde{\tau}(t)}^n(x) = \lim_{m \rightarrow \infty} \sum_{r \leq s_*^n(t)} \int_0^{\xi(\mathbf{e}_r^\uparrow)} f_{\varepsilon_m, x}(\mathbf{e}_r^\uparrow(u)) du, \text{ a.s., for all } x \in \tilde{T}_{\leq n-|a|}^a \setminus \{a\}, \quad (\text{A.6})$$

$$\begin{aligned} \tilde{L}_{\tilde{\tau}(t)}^n(y) &= \lim_{m \rightarrow \infty} \left( \int_0^{H_a} f_{\varepsilon_m, y}(\tilde{X}_u^0) du + \sum_{r < s_*^n(t)} \int_0^{\xi(\mathbf{e}_r^\uparrow)} f_{\varepsilon_m, y}(\mathbf{e}_r^\uparrow(u)) du \right. \\ &\quad \left. + \int_0^{\tilde{\tau}(t) - J^\uparrow(s_*^n(t) -)} f_{\varepsilon_m, y}(\mathbf{e}_{s_*^n(t)}^\uparrow(u)) du \right), \text{ a.s., for all } y \in \tilde{T}_{\leq n} \setminus \tilde{T}_{\leq n-|a|}^a, \end{aligned} \quad (\text{A.7})$$

where “a.s.” in (A.4)-(A.7) means “ $\tilde{P}_\rho$ -almost surely”. Let  $Y^\downarrow$  be the process defined by replacing  $J(\cdot)$  and  $\mathbf{e}$  in (A.1) with  $J^\downarrow(\cdot)$  and  $\mathbf{e}^\downarrow$ , respectively. Let  $\tilde{X}^\downarrow$  be the process defined by replacing  $Y$  with  $Y^\downarrow$  in (A.2). Then, by [26, Theorem 4.1 and Corollary 5.1], the process  $\tilde{X}^\downarrow = (\tilde{X}_t^\downarrow, t \geq 0, \tilde{P}_x, x \in \tilde{T}_{\leq n-|a|}^a)$  is a Brownian motion on  $\tilde{T}_{\leq n-|a|}^a$ . Thus, by (A.4)-(A.7) and the independence of  $\mathbf{e}^\uparrow$  and  $\mathbf{e}^\downarrow$ , together with [35, Theorem 3.6.3], the law of  $\{\tilde{L}_{\tilde{\tau}(t)}^n(x) : x \in \tilde{T}_{\leq n-|a|}^a \setminus \{a\}\}$  under  $\tilde{P}_\rho(\cdot | \mathcal{F}^\uparrow)$  is the same as that of  $\left\{ \tilde{L}_{\tilde{\tau}^\downarrow(\tilde{L}_{\tilde{\tau}(t)}^n(a))}^\downarrow(x) : x \in \tilde{T}_{\leq n-|a|}^a \setminus \{a\} \right\}$  under  $\tilde{P}_a$ .  $\square$

## A.2 Proof of Lemma 2.2

*Proof of Lemma 2.2.* Let  $\{z_1, z_2\}, \{w_1, w_2\}$  be edges of  $T$  with  $x \in I_{\{z_1, z_2\}}$  and  $y \in I_{\{w_1, w_2\}}$ . We have

$$\mathbb{E}(\tilde{h}_x \tilde{h}_y) = \tilde{E}_x(\tilde{L}_{H_\rho}^n(y))$$



$$\begin{aligned}
&= \tilde{E}_x \left( \tilde{L}_{H_{z_1} \wedge H_{z_2}}^n(y) + \tilde{L}_{H_\rho}^n(y) \circ \theta_{H_{z_1} \wedge H_{z_2}} \right) \\
&= \tilde{E}_x(\tilde{L}_{H_{z_1} \wedge H_{z_2}}^n(y)) + \tilde{P}_x(H_{z_1} < H_{z_2}) \tilde{E}_{z_1}(\tilde{L}_{H_\rho}^n(y)) + \tilde{P}_x(H_{z_2} < H_{z_1}) \tilde{E}_{z_2}(\tilde{L}_{H_\rho}^n(y)) \\
&= \tilde{E}_x(\tilde{L}_{H_{z_1} \wedge H_{z_2}}^n(y)) + \frac{d(x, z_2)}{d(z_1, z_2)} \tilde{E}_y(\tilde{L}_{H_\rho}^n(z_1)) + \frac{d(x, z_1)}{d(z_1, z_2)} \tilde{E}_y(\tilde{L}_{H_\rho}^n(z_2)). \tag{A.8}
\end{aligned}$$

Similar arguments implies that for each  $i \in \{1, 2\}$ ,

$$\tilde{E}_y(\tilde{L}_{H_\rho}^n(z_i)) = \frac{d(y, w_2)}{d(w_1, w_2)} \tilde{E}_{w_1}(\tilde{L}_{H_\rho}^n(z_i)) + \frac{d(y, w_1)}{d(w_1, w_2)} \tilde{E}_{w_2}(\tilde{L}_{H_\rho}^n(z_i)). \tag{A.9}$$

(In (A.9), we have used the equality  $\tilde{E}_y(\tilde{L}_{H_{w_1} \wedge H_{w_2}}^n(z_i)) = 0$  for each  $i \in \{1, 2\}$ .) Note that by (2.3) and [19, Lemma 2.1], for all  $i, j \in \{1, 2\}$ ,

$$\tilde{E}_{w_i}(\tilde{L}_{H_\rho}^n(z_j)) = E_{w_i}(L_{H_\rho}^n(z_j)) = d(\rho, w_i) + d(\rho, z_j) - d(w_i, z_j). \tag{A.10}$$

We have

$$\tilde{E}_x(\tilde{L}_{H_{z_1} \wedge H_{z_2}}^n(y)) = 0, \quad \text{if } I_{\{z_1, z_2\}} \neq I_{\{w_1, w_2\}}. \tag{A.11}$$

Assume that  $I_{\{z_1, z_2\}} = I_{\{w_1, w_2\}}$ . Note that a Brownian motion on  $\tilde{T}_{\leq n}$  starting at  $x$  killed upon  $H_{z_1} \wedge H_{z_2}$  has the same law as a standard Brownian motion on  $(0, \frac{1}{2})$  starting at  $d(x, z_1)$  killed upon  $H_0 \wedge H_{1/2}$ . Let  $\{L_t^B(x) : t \geq 0\}$  be a local time at  $x$  of a standard Brownian motion  $\{B_t, t \geq 0, P_y^B, y \in \mathbb{R}\}$  on  $\mathbb{R}$ . By this and [36, Chapter VI, Exercise 2.8], we have

$$\begin{aligned}
\tilde{E}_x(\tilde{L}_{H_{z_1} \wedge H_{z_2}}^n(y)) &= E_{d(x, z_1)}^B(L_{H_0 \wedge H_{1/2}}^B(d(y, z_1))) \\
&= \frac{d(x, z_2)}{d(z_1, z_2)} d(y, z_1) + \frac{d(x, z_1)}{d(z_1, z_2)} d(y, z_2) - d(x, y), \quad \text{if } I_{\{z_1, z_2\}} = I_{\{w_1, w_2\}}.
\end{aligned} \tag{A.12}$$

Thus, by (A.8)-(A.12), we have

$$\mathbb{E}(\tilde{h}_x \tilde{h}_y) = d(\rho, x) + d(\rho, y) - d(x, y). \quad \square$$

### A.3 Tail of maximum of local time revisited

In the proof of Proposition 4.1, we need a version of Proposition 3.1(ii):

**Proposition A.1** *There exist  $c_1 > 0$  and  $t_0 > 0$  such that for all  $n \in \mathbb{N}$ ,  $t \geq t_0$ , and  $y \in [0, 2\sqrt{n}]$ ,*

$$P_\rho \left( \max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \geq c_1 e^{-2\sqrt{\log b} y}. \tag{A.13}$$

**Remark A.2** *The assumption of  $t$  in Proposition A.1 is weaker than that of Proposition 3.1(ii). This is the main requirement in the proof of Proposition 4.1.*

Fix  $\varepsilon \in (0, 1/4)$ . For  $n \in \mathbb{N}$ ,  $t > 0$ ,  $y > 0$ ,  $0 < r < n$ , and  $v \in T_n$ , set

$$A_{v,r}^n(t) := \left\{ \begin{aligned} &\sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \in \sqrt{t} + \ell_{y,t,n}(s) + [-g_n(s), -f_n(s)], \text{ for all } r \leq s \leq n-r, \\ &\sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v)} \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1), \\ &\sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_{s'})} \geq \sqrt{t} + \ell_{y,t,n}(s') - r^{1/2+2\varepsilon}, \text{ for all } s' \in [0, r] \cup [n-r, n] \end{aligned} \right\},$$

where

$$\ell_{y,t,n}(s) := \left( \frac{a_n(t)}{n} + \frac{y}{n} \right) s, \quad 0 \leq s \leq n,$$

$$f_n(s) := \min \left\{ s^{1/2-\varepsilon}, (n-s)^{1/2-\varepsilon} \right\}, \quad g_n(s) := \min \left\{ s^{1/2+\varepsilon}, (n-s)^{1/2+\varepsilon} \right\}.$$

To prove Proposition A.1, we will apply the second moment method to  $\sum_{v \in T_n} 1_{A_{v,r}^n(t)}$ . We first need the following:

**Lemma A.3** *There exist  $r_0 \in \mathbb{N}$ ,  $c_1 = c_1(r_0) > 0$  such that for all  $r_0 \leq r \leq n/4$ ,  $t > 4r^{1+4\varepsilon}$ ,  $y \in [0, 2\sqrt{n}]$ , and  $v \in T_n$ ,*

$$\tilde{P}_\rho(A_{v,r}^n(t)) \geq c_1 b^{-n} e^{-2\sqrt{\log b} y}. \quad (\text{A.14})$$

*Proof.* Fix  $r_0 \leq r \leq n/4$ ,  $t > 4r^{1+4\varepsilon}$ , and  $y \in [0, 2\sqrt{n}]$ , where we take  $r_0 \in \mathbb{N}$  large enough. By Lemma 2.5 and (2.7), we have

$$\begin{aligned} &\sqrt{\frac{\sqrt{t} + a_n(t) + y + 1}{\sqrt{t}}} \tilde{P}_\rho(A_{v,r}^n(t)) \\ &\geq c_1(r_0) P_0^B \left( \begin{aligned} &\ell_{y,t,n}(s) - g_n(s) \leq B_s/\sqrt{2} < \ell_{y,t,n}(s) - f_n(s), \text{ for all } r \leq s \leq n-r, \\ &B_n/\sqrt{2} \in [a_n(t) + y, a_n(t) + y + 1), \\ &B_{s'}/\sqrt{2} \geq \ell_{y,t,n}(s') - r^{1/2+2\varepsilon}, \text{ for all } s' \in [0, r] \cup [n-r, n] \end{aligned} \right), \end{aligned} \quad (\text{A.15})$$

where we have used the following: under the event that  $\sqrt{X_s/2} \geq \sqrt{t} + \ell_{y,t,n}(s) - g_n(s)$  for all  $s \in [r, n-r]$  and  $\sqrt{X_{s'}/2} \geq \sqrt{t} + \ell_{y,t,n}(s') - r^{1/2+2\varepsilon}$  for all  $s' \in [0, r] \cup [n-r, n]$ , we have  $\exp\left(-\frac{3}{8} \int_0^n \frac{ds}{X_s}\right) \geq c_1(r_0)$ . Since  $\{B_s - B_n \frac{s}{n} : 0 \leq s \leq n\}$  is independent of  $B_n$  and has the same law as that of a standard Brownian bridge from 0 to 0 on  $[0, n]$ , the right-hand side of (A.15) is bounded from below by

$$c_1(r_0) P_{0 \rightarrow 0}^n \left( -g_n(s) \leq X_s/\sqrt{2} < -f_n(s) - 1, \text{ for all } r \leq s \leq n-r, \right)$$

$$\begin{aligned}
& \left. X_{s'}/\sqrt{2} \geq -r^{1/2+2\varepsilon}, \text{ for all } s' \in [0, r] \cup [n-r, n] \right) \\
& \cdot P_0^B \left( B_n/\sqrt{2} \in [a_n(t) + y, a_n(t) + y + 1] \right), \tag{A.16}
\end{aligned}$$

where for  $T > 0$  and  $p, q \in \mathbb{R}$ ,  $P_{p \rightarrow q}^T$  is a probability law on  $(C[0, T], \mathcal{B}(C[0, T]))$  ( $C[0, T]$  is the space of all continuous functions on  $[0, T]$  and  $\mathcal{B}(C[0, T])$  is the  $\sigma$ -field generated by cylinder sets in  $C[0, T]$ ) under which the coordinate process  $\{X_s : 0 \leq s \leq T\}$  is a standard Brownian bridge from  $p$  to  $q$  on  $[0, T]$ . We have

$$\begin{aligned}
& P_{0 \rightarrow 0}^n \left( -g_n(s) \leq X_s/\sqrt{2} < -f_n(s) - 1, \text{ for all } r \leq s \leq n-r, \right. \\
& \quad \left. X_{s'}/\sqrt{2} \geq -r^{1/2+2\varepsilon}, \text{ for all } s' \in [0, r] \cup [n-r, n] \right) \\
& = E_{0 \rightarrow 0}^n \left[ 1_{\{-g_n(r) \leq X_r/\sqrt{2} < -f_n(r) - 1, -g_n(n-r) \leq X_{n-r}/\sqrt{2} < -f_n(n-r) - 1\}} \right. \\
& \quad \cdot P_{X_r \rightarrow X_{n-r}}^{n-2r} \left( -g_n(s+r) \leq X_s/\sqrt{2} < -f_n(s+r) - 1, \text{ for all } 0 \leq s \leq n-2r \right) \\
& \quad \cdot P_{0 \rightarrow X_r}^r \left( X_s/\sqrt{2} \geq -r^{1/2+2\varepsilon}, \text{ for all } 0 \leq s \leq r \right) \\
& \quad \cdot P_{X_{n-r} \rightarrow 0}^r \left( X_s/\sqrt{2} \geq -r^{1/2+2\varepsilon}, \text{ for all } 0 \leq s \leq r \right) \Big] \\
& \geq c_2(r_0) P_{0 \rightarrow 0}^n \left( -g_n(s) \leq X_s/\sqrt{2} < -f_n(s) - 1, \text{ for all } r \leq s \leq n-r \right), \tag{A.17}
\end{aligned}$$

where we have used [15, Lemma 2.2(a)] in the second inequality which implies that the last two probabilities in the expectation in the second display of (A.17) are bounded from below by constants depending on  $r_0$ . We have for  $r_0 \leq r < n/4$  with  $r_0$  large enough

$$\begin{aligned}
& P_{0 \rightarrow 0}^n \left( -g_n(s) \leq X_s/\sqrt{2} < -f_n(s) - 1, \text{ for all } r \leq s \leq n-r \right) \\
& = P_{0 \rightarrow 0}^n \left( \sqrt{2}(f_n(s) + 1) \leq X_s \leq \sqrt{2}g_n(s), \text{ for all } r \leq s \leq n-r \right) \\
& \geq P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n-r) \\
& \quad - P_{0 \rightarrow 0}^n \left( X_s \geq 0, \text{ for all } r \leq s \leq n-r, X_{s'} > \sqrt{2}g_n(s') \text{ for some } s' \in [r, n-r] \right) \\
& \quad - P_{0 \rightarrow 0}^n \left( X_s \geq 0, \text{ for all } r \leq s \leq n-r, X_{s'} < 2f_n(s') \text{ for some } s' \in [r, n-r] \right) \\
& \geq P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n-r) \\
& \quad - \frac{1}{4} P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n-r) - \frac{1}{4} P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n-r) \\
& \geq \frac{1}{2} P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n-r, X_r, X_{n-r} \in [\sqrt{r}/2, \sqrt{r}])
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} E_{0 \rightarrow 0}^n \left[ 1_{\{X_r, X_{n-r} \in [\sqrt{r}/2, \sqrt{r}]\}} P_{X_r \rightarrow X_{n-r}}^{n-2r} (X_s \geq 0, \text{ for all } 0 \leq s \leq n-2r) \right] \\
&\geq \frac{c_3(r_0)}{n} P_{0 \rightarrow 0}^n (X_r, X_{n-r} \in [\sqrt{r}/2, \sqrt{r}]) \geq \frac{c_4(r_0)}{n},
\end{aligned} \tag{A.18}$$

where we have used the symmetry of a Brownian bridge in the first equality, [15, Lemma 2.7 and Proposition 6.1] in the third inequality, and [15, Lemma 2.2(a)] in the 6th inequality. Thus, by (A.15)-(A.18), we have (A.14).  $\square$

Next, we need the following:

**Lemma A.4** (i) *There exist  $r_0 \in \mathbb{N}$  and  $c_1 > 0$  such that for all  $t > 0$ ,  $r_0 \leq r \leq n/4$ ,  $y \in [0, 2\sqrt{n}]$ ,  $k \in \{r, r+1, \dots, n-r-1\}$ ,  $v \in T_n$ , and  $(\sqrt{t} + \ell_{y,t,n}(k) - g_n(k))^2 \leq q \leq (\sqrt{t} + \ell_{y,t,n}(k) - f_n(k))^2$ ,*

$$\begin{aligned}
&\tilde{P}_{v_k} \left[ \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(q)}^\downarrow}(v_s) < \sqrt{t} + \ell_{y,t,n}(s) - f_n(s), \text{ for all } k \leq s \leq n-r, \right. \\
&\quad \left. \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(q)}^\downarrow}(v) \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \right] \\
&\leq c_1 r^{1/2+\varepsilon} \sqrt{\frac{\sqrt{t} + \ell_{y,t,n}(k) - f_n(k)}{\sqrt{t} + a_n(t) + y} \frac{g_n(k)}{(n-k-r)\sqrt{n-k}}} b^{-(n-k)} \\
&\quad \cdot e^{\frac{3 \log n}{2n}(n-k)} e^{\frac{\log(\frac{\sqrt{t}+n}{\sqrt{t}})}{2n}(n-k)} e^{-\frac{2a_n(t)}{n}y \frac{n-k}{n}} e^{-\frac{2a_n(t)}{n}f_n(k)},
\end{aligned} \tag{A.19}$$

where  $\{\tilde{L}_s^\downarrow(x) : (s, x) \in [0, \infty) \times \tilde{T}_{\leq n-k}^{v_k}\}$  is a local time of a Brownian motion on  $\tilde{T}_{\leq n-k}^{v_k}$  and  $\tilde{\tau}^\downarrow(p) := \inf\{s \geq 0 : \tilde{L}_s^\downarrow(v_k) > p\}$ .

(ii) *There exist  $r_0 \in \mathbb{N}$  and  $c_2 > 0$  such that for all  $t > 0$ ,  $r_0 \leq r \leq n/4$ ,  $y \in [0, 2\sqrt{n}]$ , and  $v \in T_n$ ,*

$$\tilde{P}(A_{v,r}^n(t)) \leq c_2 r b^{-n} e^{-2\sqrt{\log b}y}.$$

*Proof.* Let  $P_k$  be the left-hand side of (A.19). Recall the probability measure  $P_{0 \rightarrow 0}^T$  defined in the proof of Lemma A.3. By Lemma 2.5 and (2.7), we have

$$\begin{aligned}
&\sqrt{\frac{\sqrt{t} + a_n(t) + y}{\sqrt{q}}} P_k \\
&\leq P_0^B \left[ B_s/\sqrt{2} < \sqrt{t} - \sqrt{q} + \ell_{y,t,n}(k+s) - f_n(k+s), \text{ for all } 0 \leq s \leq n-k-r, \right. \\
&\quad \left. B_{n-k}/\sqrt{2} \in [\sqrt{t} - \sqrt{q} + a_n(t) + y, \sqrt{t} - \sqrt{q} + a_n(t) + y + 1] \right] \\
&\leq P_{0 \rightarrow 0}^{n-k} \left( X_s/\sqrt{2} < \sqrt{t} - \sqrt{q} + \ell_{y,t,n}(k+s) - \frac{s}{n-k}(\sqrt{t} - \sqrt{q} + a_n(t) + y), \right.
\end{aligned}$$

$$\cdot P_0^B \left( B_{n-k}/\sqrt{2} \in [\sqrt{t} - \sqrt{q} + a_n(t) + y, \sqrt{t} - \sqrt{q} + a_n(t) + y + 1) \right), \quad (\text{A.20})$$

where we have used the fact that the process  $\{B_s - \frac{s}{n-k} \cdot B_{n-k} : 0 \leq s \leq n-k\}$  is independent of  $B_{n-k}$  and has the same law as that of a standard Brownian bridge from 0 to 0 on  $[0, n-k]$ . We estimate two probabilities on the right-hand side of (A.20). By the assumption of  $q$ , we have

$$\begin{aligned} & P_0^B \left( B_{n-k}/\sqrt{2} \in [\sqrt{t} - \sqrt{q} + a_n(t) + y, \sqrt{t} - \sqrt{q} + a_n(t) + y + 1) \right) \\ & \leq \frac{c_1}{\sqrt{n-k}} b^{-(n-k)} e^{\frac{3 \log n}{2n}(n-k)} e^{\frac{\log(\frac{\sqrt{t}+n}{\sqrt{t}})}{2n}(n-k)} e^{-\frac{2a_n(t)}{n}y \frac{n-k}{n}} e^{-\frac{2a_n(t)}{n}f_n(k)}. \end{aligned} \quad (\text{A.21})$$

To estimate the other probability, we use [3, Lemma 3.4]: for any  $x_1, x_2 \in [0, \infty)$ ,  $r_1, r_2 \in [0, \infty)$ , and  $T > r_1 + r_2$ ,

$$\begin{aligned} & P_{0 \rightarrow 0}^T \left( X_s \leq \left(1 - \frac{s}{T}\right) x_1 + \frac{s}{T} x_2, r_1 \leq s \leq T - r_2 \right) \\ & \leq \frac{2}{T - r_1 - r_2} \left\{ \left(1 - \frac{r_1}{T}\right) x_1 + \frac{r_1}{T} x_2 + \sqrt{r_1} \right\} \left\{ \frac{r_2}{T} x_1 + \left(1 - \frac{r_2}{T}\right) x_2 + \sqrt{r_2} \right\}. \end{aligned} \quad (\text{A.22})$$

By (A.22), we have

$$\begin{aligned} & P_{0 \rightarrow 0}^{n-k} \left( \frac{X_s}{\sqrt{2}} < \sqrt{t} - \sqrt{q} + \ell_{y,t,n}(k+s) - \frac{s}{n-k}(\sqrt{t} - \sqrt{q} + a_n(t) + y) \right. \\ & \quad \left. \text{for all } 0 \leq s \leq n-k-r \right) \\ & \leq c_2 \frac{r^{1/2+\varepsilon}}{n-k-r} g_n(k). \end{aligned} \quad (\text{A.23})$$

Thus, by (A.20), (A.21), and (A.23), we have (A.19). By repeating a similar argument, we can also prove (ii). We omit the detail.  $\square$

*Proof of Proposition A.1.* Using Lemma A.3, A.4, and techniques in the proof of Lemma A.4, we can prove Proposition A.1 in almost the same way to the proof of Proposition 3.1(ii). We omit the details.  $\square$

#### A.4 Tail of maximum of BRW

In the proof of Proposition 5.1, we use tail estimates of the maximum of the BRW on  $T$ . Let  $(h_v)_{v \in T}$  be a BRW on  $T$  defined in Section 1.

**Lemma A.5** (i) *There exist  $c_1, c_2 \in (0, \infty)$  such that for all  $y > 0$  and  $n \in \mathbb{N}$ ,*

$$\mathbb{P} \left( \max_{v \in T_n} h_v / \sqrt{2} > \sqrt{\log b} n - \frac{3}{4\sqrt{\log b}} \log n + y \right) \leq c_1 (1+y) e^{-2\sqrt{\log b} y} e^{-c_2 \frac{y^2}{n}}.$$

(ii) There exists  $c_3 > 0$  and  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  and  $y \in [1, \sqrt{n}]$ ,

$$\mathbb{P} \left( \max_{v \in T_n} h_v / \sqrt{2} > \sqrt{\log b} n - \frac{3}{4\sqrt{\log b}} \log n + y \right) \geq c_3 y e^{-2\sqrt{\log b} y}.$$

Lemma A.5(ii) is a special version of Lemma 2.7 of [17]. One can easily modify the proof of Lemma 3.8 in [16] (this is basically a tail estimate of the maximum of a BRW on a 4-ary tree) to prove Lemma A.5(i). We omit the details.

### Acknowledgments.

The author would like to thank Professor Kumagai for valuable comments and encouragement, and Professor Biskup for stimulating discussion and pointing out a connection between my work and the theory of random multiplicative cascade measures. This work is partially supported by JSPS KAKENHI 13J01411 and University Grants for student exchange between universities in partnership under Top Global University Project of Kyoto University.

## References

- [1] E. Aïdékon. Convergence in law of the minimum of a branching random walk. *Ann. Probab.* **41** (2013), 1362-1426.
- [2] E. Aïdékon, J. Berestycki, É. Brunet and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Relat. Fields.* **157** (2013), 405-451.
- [3] L.-P. Arguin, A. Bovier, and N. Kistler. Genealogy of extremal particles of branching Brownian motion. *Commun. Pure Appl. Math.* **64** (2011), 1647-1676.
- [4] L.-P. Arguin, A. Bovier, and N. Kistler. Poissonian statistics in the extremal process of branching Brownian motion. *Ann. Appl. Probab.* **22** (2012), 1693-1711.
- [5] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Relat. Fields.* **157** (2013), 535-574.
- [6] M. Bachmann. Limit theorems for the minimal position in a branching random walk with independent logconcave displacements. *Adv. in Appl. Probab.* **32** (2000), 159-176.
- [7] J. Barral, R. Rhodes, and V. Vargas. Limiting laws of supercritical branching random walks. *C. R. Acad. Sci. Paris, Ser. I.* **350** (2012), 535-538.
- [8] J. Barral, A. Kupiainen, M. Nikula, E. Saksman, and C. Webb. Critical Mandelbrot cascades. *Commun. Math. Phys.* **325** (2014), 685-711.
- [9] D. Belius and N. Kistler. The subleading order of two dimensional cover times. arXiv:1405.0888v1.
- [10] J. D. Biggins and A. E. Kyprianou. Measure change in multitype branching. *Adv. Appl. Prob.* **36** (2004), 544-581.

- [11] M. Biskup and O. Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *Commun. Math. Phys.* (to appear)
- [12] M. Biskup and O. Louidor. Conformal symmetries in the extremal process of two-dimensional discrete Gaussian free field. arXiv:1410.4676
- [13] A. Bovier and L. Hartung. Extended convergence of the extremal process of branching Brownian motion. arXiv:1412.5975
- [14] K. Bogus and J. Małęcki. Sharp estimates of transition probability density for Bessel process in half-line. *Potential Anal.* **43** (2015), 1-22.
- [15] M. Bramson. Convergence of solutions of the Kolmogorov equation to traveling waves. *Mem. Amer. Math. Soc.* **44** (1983)
- [16] M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Commun. Pure Appl. Math.* **69** (2016), 62-123.
- [17] M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of nonlattice branching random walk. arXiv:1404.3423.
- [18] J. Ding. Asymptotics of cover times via Gaussian free fields: Bounded-degree graphs and general trees. *Ann. Probab.* **42** (2014), 464-496.
- [19] J. Ding, J. R. Lee, and Y. Peres. Cover times, blanket times, and majorizing measures. *Ann. of Math.* **175** (2012), 1409-1471.
- [20] D. J. Daley and D. Vere-Jones. An Introduction to the Theory of Point Processes. Volume II: General Theory and Structure. Second Edition. Springer, 2008.
- [21] J. Ding and O. Zeitouni. A sharp estimate for cover times on binary trees. *Stochastic Processes and their Applications.* **122** (2012), 2117-2133.
- [22] J. Ding and O. Zeitouni. Extreme values for two-dimensional discrete Gaussian free field. *Ann. Probab.* **42** (2014), 1480-1515.
- [23] E. B. Dynkin. Markov processes as a tool in field theory. *J. Funct. Anal.*, **50** (1983), 167-187.
- [24] N. Eisenbaum, H. Kaspi, M. B. Marcus, J. Rosen, and Z. Shi. A Ray-Knight theorem for symmetric Markov processes. *Ann. Probab.* **28** (2000), 1781-1796.
- [25] M. Folz. Volume growth and stochastic completeness of graphs. *Trans. Amer. Math. Soc.*, **366** (2014), 2089-2119.
- [26] M. Fukushima and H. Tanaka. Poisson point processes attached to symmetric diffusions. *Ann. Inst. Henri. Poincaré Probab. Stat.*, **41**, (2005), 419-459.
- [27] S. Haeseler. Heat kernels estimates and related inequalities on metric graphs. arXiv:1101.3010v1.

- [28] I. Karatzas and S. E. Shreve. Brownian Motion and Stochastic Calculus, 2nd ed. Graduate Texts in Mathematics., vol. 113, Springer-Verlag, New York.-Berlin, 1991.
- [29] V. Kostykin, J. Potthoff, and R. Schrader. Brownian motion on metric graphs. *J. Math. Phys.* **53** (2012)
- [30] S.P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.* **15** (1987), 1052-1061.
- [31] T. Lupu. From loop clusters and random interlacements to the free field. arXiv:1402.0298.
- [32] R. Lyons. A simple path to Biggins' martingale convergence for branching random walk. In *Classical and Modern Branching Processes (Minneapolis, MN, 1994)*. IMA Vol. Math. Appl. **84** 217-221. Springer, New York.
- [33] T. Madaule. Convergence in law for the branching random walk seen from its tip. *J. Theor. Probab.* (to appear)
- [34] M. B. Marcus and J. Rosen. Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes. *Ann. Probab.* **20** (1992), 1603-1684.
- [35] M. B. Marcus and J. Rosen. Markov processes, Gaussian processes and local times. Cambridge University Press, 2006.
- [36] D. Revuz and M. Yor. Continuous martingales and Brownian motion. Springer, 3rd edition, 1999.
- [37] E. Subag and O. Zeitouni. Freezing and decorated poisson point processes. *Commun. Math. Phys.* **337** (2015), 55-92.
- [38] A. Zhai. Exponential concentration of cover times. arXiv:1407.7617v1.